Exercises III

1. Using the binomial model for X, $\mathrm{Bin}(N, p)$, assume that we know p and observe X (and always $X \geq 1$), but not what the sample size N was, and that we don't know any upper limit for N. Because we assume $X \geq 1$, also $N \geq 1$. The possible values after observing X are then $N \in \{X, X+1, X+2, \ldots\}$. To describe uncertainty prior to data, choose the improper prior $\pi(N) \propto 1/N$. It is improper, because $\sum_{n=1}^{\infty} 1/n = \infty$. Solve the posterior distribution $\pi(N \mid X, p)$. (Hint: look for Negative Binomial distribution when computing the normalizing constant).

$$\pi(N \mid X, p) \propto \pi(X \mid N, p)\pi(N) = \binom{N}{X} p^X (1-p)^{N-X} \times \frac{1}{N}$$

The easiest way is to see if this, as a function of N, can be written in the form of some familiar distribution. By rewriting it:

$$\frac{1}{X}\binom{N-1}{X-1} p^X (1-p)^{N-X},$$

which is proportional to a Negative Binomial distribution for $N \geq X$. Alternatively, by solving explicitly the normalizing constant

$$\sum_{N=X}^{\infty} \binom{N}{X} p^X (1-p)^{N-X} \times 1/N = \sum_{N=X}^{\infty} \frac{N!}{X!(N-X)!} p^X (1-p)^X 1/N$$

$$= \sum_{N=X}^{\infty} \frac{1}{X} \frac{(N-1)!}{(X-1)!(N-1-(X-1))!} p^X (1-p)^{N-X} = \frac{1}{X} \sum_{N=X}^{\infty} \underbrace{\binom{N-1}{X-1} p^X (1-p)^{N-X}}_{\text{NegBinDistrib for } N} = \frac{1}{X}$$

The posterior is then (a proper distribution, even though prior was improper)

$$\pi(N \mid X, p) = \frac{X}{N}\binom{N}{X} p^X (1-p)^{N-X} = \binom{N-1}{X-1} p^X (1-p)^{N-X} = \mathrm{NegBin}(X, p)$$

[Note: there are different parameterizations of Negative Binomial distribution in different books, leading to seemingly different expressions].

2. Let $\pi(X \mid N, p) = \mathrm{Bin}(N, p)$ and $\pi(p) = \mathrm{U}(0, 1)$. Show that the prior predictive distribution of X is discrete uniform $\pi(X) = 1/(N+1)$, $\forall X \in \{0, 1, 2, \ldots, N\}$.

$$\pi(X) = \int_0^1 \binom{N}{X} p^X (1-p)^{N-X} \mathbf{d}p$$

$$= \binom{N}{X} \int_0^1 \frac{\Gamma(X+1)\Gamma(N+2-(X+1))}{\Gamma(N+2)} \underbrace{\frac{\Gamma(N+2)}{\Gamma(X+1)\Gamma(N+2-(X+1))} p^{X+1-1}(1-p)^{N-X+1-1}}_{\mathrm{Beta}(X+1, N-X+1)} \mathbf{d}p$$

$$= \binom{N}{X} \frac{X!(N-X)!}{(N+1)N!} = \frac{1}{N+1}$$

3. Compute the variance of beta-binomial distribution, using the law of total variance.

$$V(X) = E(V(X \mid r, N)) + V(E(X \mid r, N))$$

$$= E(Nr(1-r)) + V(rN) = N(E(r) - \underbrace{E(r^2)}_{V(r)+E(r)^2}) + N^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= N\Big(\frac{\alpha}{\alpha+\beta} - \big(\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2}\big)\Big) + N^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= N\frac{\alpha(\alpha+\beta)(\alpha+\beta+1) - \alpha\beta - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} + N^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{N\alpha\beta(\alpha+\beta) + N^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{N\alpha\beta(\alpha+\beta+N)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

4. Show that if we define $\psi = \log(p/(1-p))$ for $p \in (0,1)$, and uniform improper prior $\pi(\psi) \propto 1$, then this leads to Haldane's improper prior Beta$(0,0)$ for p. (Hint: use transform of variables rule). Let X be binomially distributed as Bin(N,p). Why the posterior distribution of p is not proper density if $X = 0$ or $X = N$?

$$\pi(p) = \pi(\psi(p)) \mid \frac{\mathbf{d}\psi(p)}{\mathbf{d}p} \mid = \frac{1-p}{p(1-p)^2} = p^{-1}(1-p)^{-1}$$

With $X = 0$ or $X = N$ the posterior of p is proportional to $p^{-1}(1-p)^{N-1}$ or $p^{N-1}(1-p)^{-1}$ which goes to infinity when approaching 0 or 1.

5. Give an example of a $\pi(p) = $ Beta(α,β) prior distribution and data X from Bin(N,p) where the posterior variance is larger than prior variance.

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This can happen with small values. For example, trying some in R:
a<-6;b<-1; X<-1; N<-7; A<-X+a; B<-N-X+b;
v<-c(a*b/((a+b+1)*(a+b)^2),A*B/((A+B+1)*(A+B)^2)); v
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