Exercises I

1. Jack survived a shipwreck and drifted to an island which is populated with probability p. If it's unpopulated he survives with probability p_0. If it's populated, the inhabitants are cannibals with probability p_1 and then his survival probability is p_2 but if they are not cannibals his survival probability is p_3. After all, Jack survived! What is the probability that the island was inhabited but not by cannibals?

$A =$ 'island is populated'. $B =$ 'he survived', $C =$ 'inhabitants are cannibals'.

$$P(AC^c \mid B) = \frac{P(AC^c)P(B \mid AC^c)}{P(B)}$$

For this we need $P(B)$, by using the law of total probability and product rule:

$$P(B) = P(A^c)P(B \mid A^c) + P(A)P(B \mid A)$$

$$= P(A^c)P(B \mid A^c) + P(A)P(C \mid A)P(B \mid C) + P(A)P(C^c \mid A)P(B \mid C^c)$$

$$= (1-p)p_0 + pp_1p_2 + p(1-p_1)p_3$$

$$\frac{P(AC^c)P(B \mid AC^c)}{P(B)} = \frac{P(A)P(C^c \mid A)P(B \mid C^c)}{(1-p)p_0 + pp_1p_2 + p(1-p_1)p_3} = \frac{p(1-p_1)p_3}{(1-p)p_0 + pp_1p_2 + p(1-p_1)p_3}$$

2. 60 cards have black front side and white back side. 20 cards have both sides black. 20 cards have both sides white. One card is drawn 'at random' and we note that one side is black. What is the probability that also the other side is black? Is the posterior probability of double black card then larger or smaller than the prior?

$$P(\text{double B} \mid \text{1st B}) =$$

$$\frac{P(\text{1st B} \mid \text{double B})P(\text{double B})}{P(\text{1st B} \mid \text{double B})P(\text{double B}) + P(\text{1st B} \mid \text{double W})P(\text{double W}) + P(\text{1st B} \mid \text{B \& W})P(\text{B \& W})}$$

$$= \frac{1 \times 20/100}{1 \times 20/100 + 0 \times 20/100 + 1/2 \times 60/100} = \frac{20}{100}\frac{100}{50} = 40\%$$

The posterior probability for double B is larger than the prior probability which was 20%.

3. Using product rule, factorize the joint density $\pi(x, y, z)$ into a product of three parts. Without assumptions of independence, in how many different ways this can be done?

$$\pi(x \mid y, z)\pi(y, z) = \pi(x \mid y, z)\pi(y \mid z)\pi(z)$$

or

$$\pi(x \mid y, z)\pi(z \mid y)\pi(y)$$

or

$$\pi(y \mid x, z)\pi(x \mid z)\pi(z)$$

or

$$\pi(y \mid x, z)\pi(z \mid x)\pi(x)$$

or

$$\pi(z \mid x, y)\pi(x \mid y)\pi(y)$$

or

$$\pi(z \mid x, y)\pi(y \mid x)\pi(x)$$

4. Subjective beliefs can be subject to psychological fallacies: the most often-cited example of conjunction fallacy originated with Amos Tversky and Daniel Kahneman:
'Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.'

Which of the following two statements is more probable than the other and why? (1) Linda is a bank teller. (2) Linda is a bank teller and is active in the feminist movement.

According to psychology, many would respond that (2) is more probable. However, this is not logical because (2) is a subset of (1): for every individual for which (2) is true, also (1) is true, but not the reverse. $P(A\&B \mid C) \leq P(A \mid C)$.

5. As we have seen, Bayes formula follows from the product rule. Show that the general sum rule $P(A \text{ or } B) = P(A) + P(B) - P(A\&B)$ can be obtained from the product rule $P(A\&B) = P(A \mid B)P(B) = P(B \mid A)P(A)$ and the rule of negation $P(A) = 1 - P(A^c)$. Hint: start by writing: $P(A \text{ or } B) = 1 - P(A^c\&B^c)$.

$$P(A \text{ or } B) = 1 - P(A^c\&B^c) = 1 - P(A^c)P(B^c \mid A^c) = 1 - P(A^c)(1 - P(B \mid A^c))$$

$$= P(A) + P(A^cB) = P(A) + P(B)P(A^c \mid B) = P(A) + P(B)(1 - P(A \mid B)) = P(A) + P(B) - P(A\&B)$$

6. X has uniform probability density over $(0, 1]$, ($\pi(x) = 1$ if $x \in (0, 1]$ and zero elsewhere). What is the probability density of $Y = \ln(X)$? Show that it integrates to one.

$$\pi_Y(y) = \pi_X(x(y)) \mid \frac{\mathbf{d}x(y)}{\mathbf{d}y} \mid = 1 \times e^y \quad \text{for } y \in (-\infty, 0].$$

$$\int_{-\infty}^{0} e^y \mathbf{d}y = e^0 - e^{-\infty} = 1$$