1. Merkitään $r(\bar{x}) = |\bar{x}-\bar{y}| = \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{1/2}$

$$\Rightarrow \frac{\partial r}{\partial x_i} = \frac{1}{\cancel{2}} \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{-\frac{1}{2}} \cdot \underbrace{\cancel{2}(x_i - y_i)}_{\text{sisäfunktion derivaatta}}$$

$$\left(\frac{\partial}{\partial x_i} (x_i^2 - 2x_i y_i - y_i^2) \right)$$

$$= \frac{x_i - y_i}{|\bar{x}-\bar{y}|}.$$

$$\Rightarrow \nabla \left(-\log |\bar{x}-\bar{y}| \right) = -\frac{1}{|\bar{x}-\bar{y}|} \frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}$$

$$= -\frac{1}{|\bar{x}-\bar{y}|^2} (\bar{x}-\bar{y})$$

$\Rightarrow$ kun $\bar{x}, \bar{y} \in \mathbb{R}^2$, $\bar{x} \neq \bar{y}$

$$\Delta \left(-\log |\bar{x}-\bar{y}| \right) = \nabla \cdot \nabla \left(-\log |\bar{x}-\bar{y}| \right)$$

$$= \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\frac{-(x_i - y_i)}{(x_1-y_1)^2 + (x_2-y_2)^2} \right)$$

$$= 2 \left(\frac{-1}{|\bar{x}-\bar{y}|^2} \right) + \sum_{i=1}^{2} -(-1) \frac{(x_i - y_i) \cdot 2(x_i - y_i)}{\left((x_1-y_1)^2 + (x_2-y_2)^2 \right)^2}$$

$$= \frac{-2}{|\bar{x}-\bar{y}|^2} + 2 \left(\frac{(x_1-y_1)^2 + (x_2-y_2)^2}{\left((x_1-y_1)^2 + (x_2-y_2)^2 \right)^2} \right)$$

$$= \frac{-2}{|\bar{x}-\bar{y}|^2} + 2 \frac{|\bar{x}-\bar{y}|^{\cancel{2}}}{|\bar{x}-\bar{y}|^{\cancel{4}\,2}} = \frac{-2+2}{|\bar{x}-\bar{y}|^2} \equiv 0$$

$\longrightarrow$

Jos tans $\bar{x}, \bar{y} \in \mathbb{R}^n$, $n \geq 3$, $\bar{x} \neq \bar{y}$, 5 V 2 7

$$\frac{\partial}{\partial x_i} |\bar{x}-\bar{y}|^{2-n} = \frac{\partial}{\partial x_i} \left(\left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{\frac{2-n}{2}} \right)$$

$$= \frac{2-n}{\cancel{2}} \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{\frac{2-n}{2} - 1} \cdot \cancel{2} (x_i - y_i)$$

$$= (2-n) \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{\frac{-n}{2}} (x_i - y_i)$$

$$\Rightarrow \nabla |\bar{x}-\bar{y}|^{2-n} = \frac{(2-n)}{|x-y|^n} (\bar{x}-\bar{y})$$

$$\Rightarrow \Delta |\bar{x}-\bar{y}|^{2-n} = \nabla \cdot \nabla |\bar{x}-\bar{y}|^{2-n}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left((2-n) \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{\frac{-n}{2}} (x_i - y_i) \right)$$

$$= n(2-n) \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{\frac{-n}{2}} + \sum_{k=1}^{n} \frac{-n}{\cancel{2}} (2-n) \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{-\frac{(n+2)}{2}} \cancel{2} (x_i - y_i)^2$$

$$= \frac{n(2-n)}{|\bar{x}-\bar{y}|^n} - n(2-n) \left(\frac{|\bar{x}-\bar{y}|^{\cancel{2}}}{|\bar{x}-\bar{y}|^{n+\cancel{2}}} \right)$$

$$= \frac{n(2-n) - n(2-n)}{|\bar{x}-\bar{y}|^n} = 0.$$

$\square$

2.

a) $\frac{\partial}{\partial x} e^{x^2-y^2} = 2x e^{x^2-y^2}$

ja $\frac{\partial}{\partial x}\left(2xe^{x^2-y^2}\right) = 2e^{x^2-y^2} + 2x \cdot 2x e^{x^2-y^2}$

Samalla tavalla $\frac{\partial}{\partial y}$:t (Huomaa -1).

$\Rightarrow$ $\Delta\left(e^{x^2-y^2}\right) = (2-2)e^{x^2-y^2} + 2(x^2+y^2)e^{x^2-y^2}$

$= 2(x^2+y^2)e^{x^2-y^2}$

mikä ei ole identtisesti nolla missään $\mathbb{R}^2$:n avoimessa joukossa.

$\Rightarrow$ $e^{x^2-y^2}$ ei ole harmoninen missään $\mathbb{R}^2$:n alueessa.

b) $\frac{\partial}{\partial x}\left(\sin(5x)e^{5y}\right) = 5\cos(5x)e^{5y}$

ja $\frac{\partial}{\partial x}\left(5\cos(5x)e^{5y}\right) = -5^2\sin(5x)e^{5y}$.

$\frac{\partial}{\partial y}\left(\sin(5x)e^{5y}\right) = 5\sin(5x)e^{5y}$

ja $\frac{\partial}{\partial y}\left(\sin(5x)e^{5y}\right) = 5^2\sin(5x)e^{5y}$

$\Rightarrow$ $\Delta\left(\sin(5x)e^{5y}\right) = (-25+25)\sin(5x)e^{5y} \equiv 0$

$\Rightarrow$ $\sin(5x)e^{5y}$ on harmoninen koko $\mathbb{R}^2$:ssa.

c) $$\frac{\partial}{\partial x}\left(\frac{x}{(x^2-y^2)}\right) = \frac{1}{(x^2-y^2)} - \frac{x \cdot 2x}{(x^2-y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{(x^2-y^2)} - \frac{2x^2}{(x^2-y^2)^2}\right) = -\frac{2x}{(x^2-y^2)^2} - \frac{4x}{(x^2-y^2)^2}$$

$$-(-2)\frac{2x^2 \cdot 2x}{(x^2-y^2)^3} = \frac{-6x}{(x^2-y^2)^2} + \frac{8x^3}{(x^2-y^2)^3}.$$

$$\frac{\partial}{\partial y}\left(\frac{x}{(x^2-y^2)}\right) = \frac{-2xy}{(x^2-y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{-2xy}{(x^2-y^2)^2}\right) = \frac{-2x}{(x^2-y^2)^2} - (-1)\frac{2xy \cdot 2y}{(x^2-y^2)^3}$$

$$= \frac{-2x}{(x^2-y^2)^2} + \frac{4xy^2}{(x^2-y^2)^3}$$

$$\Rightarrow \Delta \frac{x}{(x^2-y^2)} = \frac{-6x-2x}{(x^2-y^2)^2} + \frac{8x^3+4xy^2}{(x^2-y^2)^3}$$

$$= \frac{-8x(x^2-y^2)+8x^3+4xy^2}{(x^2-y^2)^3}$$

$$= \frac{\cancel{-8x^3}+8xy^2\cancel{+8x^3}+4xy^2}{(x^2-y^2)^3} = \frac{12xy^2}{(x^2-y^2)^3}$$

mikä ei ole identtisesti nolla missään $\mathbb{R}^2$:n avoimessa joukossa

$\Rightarrow \frac{x}{(x^2-y^2)}$ ei ole harmoninen missään $\mathbb{R}^2$:n alueessa.

$\longrightarrow$

d) $\frac{\partial}{\partial x} x^{10} = 10x^9$ ja $\frac{\partial}{\partial x} 10x^9 = 90x^8$

$\frac{\partial}{\partial y} x^{10} = 0$ ja $\frac{\partial}{\partial y}(0) = 0$

$\Rightarrow \Delta x^{10} = 90x^8$, mikä ei ole identtisesti nolla missään alueessa $U \subset \mathbb{R}^2$

$\Rightarrow x^{10}$ ei ole harmoninen missään $\mathbb{R}^2$:n alueessa.

3.-4. Osoita, että jos u toteuttaa monisteen integraaliyhtälön (7.8) eli

$$u(\bar{x},t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\,d\bar{y}$$

$$+ \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/(4(t-s))} u(\bar{y},s)^2\,d\bar{y}\,ds$$

niin se toteuttaa semilineaarisen lämpöyhtälön (PAREL)-(ALKEL), luennot, sivu 39. Riittää esittää todistuksen pääkohdat, voi derivoida integraalimerkin alla ilman perusteluja.

Ratkaisu:

olkoon

$$v(\bar{x},t) := \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\, d\bar{y}$$

ja

$$w(\bar{x},t) := \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/(4(t-s))} u(\bar{y},s)^2\, d\bar{y}\, ds$$

Tällöin

$$\frac{\partial v(\bar{x},t)}{\partial x_i} = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \frac{-2(x_i-y_i)}{(4t)} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\, d\bar{y}$$

$\Rightarrow$

$$\frac{\partial^2 v(\bar{x},t)}{\partial x_i^2} = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{-2(x_i-y_i)}{4t}\right)^2 e^{-(\bar{x}-\bar{y})^2/4t} \phi(\bar{y})\, d\bar{y}$$

$$+ \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \frac{-2}{4t} e^{-(\bar{x}-\bar{y})^2/4t} \phi(\bar{y})\, d\bar{y}$$

$\Rightarrow$

$$\Delta v(\bar{x},t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{(x_i-y_i)^2}{4t^2} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\, d\bar{y}$$

$$+ \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} -\frac{n}{2t} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\, d\bar{y}$$

Derivaatta t:n suhteen:

$$v_t(\bar{x},t) = \frac{-n/2}{(4\pi)^{n/2} t^{(n-2)/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\,d\bar{y}$$

$$+ \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} (-1)\left(-\frac{(\bar{x}-\bar{y})^2}{4t^2}\right) e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\,d\bar{y}$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} -\frac{n}{2t} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\,dy$$

$$+ \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{4t^2} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\,d\bar{y}.$$

Eli: $v_t(\bar{x},t) = \Delta v(\bar{x},t)$ kaikilla $\bar{x} \in \mathbb{R}^n$, $t \geq 0$.

Tekemällä w:lle saman tapaiset laskut kuin v:lle nähdään, että

$$\Delta w(\bar{x},t) = \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{(x_i-y_i)}{4(t-s)^2} e^{-(\bar{x}-\bar{y})^2/4(t-s)} u(\bar{y},s)^2\,d\bar{y}\,ds$$

$$+ \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} -\frac{n}{2t} e^{-(\bar{x}-\bar{y})^2/4(t-s)} u(\bar{y},s)^2\,dy\,ds$$

Saman tapaisin perusteluin kuin harjoitusten 3 tehtävässä 2. nähdään, että

$$w_t(\bar{x},t) = \lim_{s\to t} \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/4(t-s)} u^2(\bar{y},s)^2 d\bar{y} ds$$

$$+ \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{(x_i-y_i)^2}{4(t-s)^2} e^{-(\bar{x}-\bar{y})^2/4(t-s)} u(\bar{y},s)^2 d\bar{y} ds$$

$$+ \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} \frac{-n}{2t} e^{-(\bar{x}-\bar{y})^2/4(t-s)} u(\bar{y},s)^2 d\bar{y} ds$$

Tästä nähdään, että

$$w_t(\bar{x},t) = \Delta w(\bar{x},t) + \lim_{s\to t} \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/4(t-s)} u(\bar{y},s)^2 d\bar{y} ds$$

Kaiken kaikkiaan saadaan siis:

$$u_t(\bar{x},t) = \Delta u(\bar{x},t) + \lim_{s\to t} \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/4(t-s)} u(\bar{y},s)^2 d\bar{y} ds$$

$$= \Delta u(\bar{x},t) + u(x,t)^2;$$

mikä nähdään soveltamalla luentomonisteen lauseen 3.7. vastinetta $\mathbb{R}^n$:ssä ja käyttämällä funktion $u(\bar{y},\bar{s})^2$ jatkuvuudesta seuraavan hajotelmaa

$u(\bar{y},s)^2 = u(\bar{y},t)^2 + \varepsilon(\bar{y},t-s)$, missä $\varepsilon(\bar{y},t-s) \to 0$, kun $s \to t$

Samalla tavalla, käyttämällä lauseen 3.7. vastinetta $\mathbb{R}^n$:ssä, nähdään että

$$\lim_{t \to 0} v(\bar{x},t) = \phi(\bar{x}) \quad \Omega$$

$$\lim_{t \to 0} w(\bar{x},t) \equiv 0, \quad \text{kun}$$

5. Samanlaiset laskut voidaan toistaa myös jos u^2 korvataan kaavassa termillä u^5 tai $u^3/(1+u^2)$. Tällöin saadaan siis

$$u(\bar{x},t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/(4t)} \phi(\bar{y})\,d\bar{y}$$
$$+ \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-(\bar{x}-\bar{y})^2/(4(t-s))} F(u(\bar{y},s))\,d\bar{y}\,ds,$$

missä

$$F(u(\bar{y},s)) = u(\bar{y},s)^5$$

tai $F(u(\bar{y},s)) = \dfrac{u(\bar{y},s)^3}{1+u(\bar{y},s)^2}$

tapauksesta riippuen.