OSITTAISDIFFERENTIAALIYHTÄLÖT
LASKUHARJOITUS 6
KEVÄT 2012

1. Osoita (suorilla laskuilla), että tapauksessa $\bar{x} := (x, y) \in \mathbf{R}^2$ aaltoyhtälö napakoordinaateissa on

$$v_{tt} = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta}, \tag{1}$$

eli jos $u = u(x, y, t)$ toteuttaa aaltoyhtälön, niin funktio $v(r, \theta, t) := u(r\cos\theta, r\sin\theta, t)$ toteuttaa yhtälön (1). Tässä $r > 0$ ja $\theta \in [0, 2\pi]$.

2.–3. Johda luentojen kaava (6.52) eli tehtävän (f annettu)

$$u_{tt} = u_{xx} + u_{yy} + f(x, y, t), \quad \bar{x} = (x, y) \in \mathbf{R}^2, t \geq 0,$$

$$u(\bar{x}, 0) = 0, \quad \bar{x} \in \mathbf{R}^2,$$

$$u_t(\bar{x}, 0) = 0, \quad \bar{x} \in \mathbf{R}^2,$$

ratkaisukaava

$$u(\bar{x}, t) := \frac{1}{2\pi} \int_0^t \int_{|\bar{\xi} - \bar{x}| \leq t - \tau} \frac{f(\bar{\xi}, \tau) d\bar{\xi} d\tau}{\sqrt{(t-\tau)^2 - |\bar{\xi} - \bar{x}|^2}}$$

4.–5. Ratkaise formaalisti Fourier–sarja–menetelmällä alkuarvo–reuna–arvotehtävä

$$u_{tt} - u_{xx} + u = 0 \quad , \quad 0 < x < \pi \ , \ t > 0,$$

$$u_x(0, t) = u_x(\pi, t) \quad , \quad t > 0,$$

$$u(x, 0) = 0 \quad , \quad 0 \leq x \leq \pi$$

$$u_t(x, 0) = 1 + \cos^3 x \quad , \quad 0 \leq x \leq \pi.$$

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1. Using straightforward calculations, show that for $\bar{x} := (x, y) \in \mathbf{R}^2$, the wave equation in polar coordinates reads as

$$v_{tt} = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta}; \tag{1}$$

in other words, if $u = u(x, y, t)$ satisfies the wave equation, then the function $v(r, \theta, t) := u(r\cos\theta, r\sin\theta, t)$ satisfies the equation (1). Here, $r > 0$ and $\theta \in [0, 2\pi]$.

2.–3. Prove the formula (6.52) of the lecture notes, or, show that the solution of the problem

$$u_{tt} = u_{xx} + u_{yy} + f(x, y, t), \quad \bar{x} = (x, y) \in \mathbf{R}^2, t \geq 0,$$

$$u(\bar{x}, 0) = 0, \quad \bar{x} \in \mathbf{R}^2,$$
$$u_t(\bar{x}, 0) = 0, \quad \bar{x} \in \mathbf{R}^2,$$

is given by

$$u(\bar{x}, t) := \frac{1}{2\pi} \int_0^t \int_{|\bar{\xi}-\bar{x}| \le t-\tau} \frac{f(\bar{\xi}, \tau) d\bar{\xi} d\tau}{\sqrt{(t-\tau)^2 - |\bar{\xi} - \bar{x}|^2}}$$

(f is known).

4.–5. Solve formally using the Fourier-series method the problem

$$u_{tt} - u_{xx} + u = 0 \quad , \quad 0 < x < \pi \ , \ t > 0,$$

$$u_x(0, t) = u_x(\pi, t) \quad , \quad t > 0,$$
$$u(x, 0) = 0 \quad , \quad 0 \le x \le \pi$$
$$u_t(x, 0) = 1 + \cos^3 x \quad , \quad 0 \le x \le \pi.$$