1 Introduction

The scope is to understand under which condition a sequence of *ε-periodic* functions $u^{(\varepsilon)}(\boldsymbol{x})$

$$u\colon \Omega \mapsto \mathbb{R} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^d$ can be approximated in the form of a series

$$u^{(\varepsilon)}(\boldsymbol{x}) = u_{(0)}\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) + \varepsilon\, u_{(1)}\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) + \varepsilon^2\, u_{(2)}\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) + \dots \tag{1.2}$$

in the limit of vanishing $\varepsilon > 0$. The main reference for the results presented in these notes is [1].

2 Convergence results for periodically oscillating functions in $\mathbb{L}^1$

Let Ω an open set in $\mathbb{R}^d$ and $Y = [0,1]^d$ the unit cube in $\mathbb{R}^d$.

Definition 2.1. *A function $\psi(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{L}^1(\Omega \times Y)$, Y-periodic in $\boldsymbol{y}$, is called an "admissible" test function if and only if*

$$\lim_{\varepsilon \downarrow 0} \int_\Omega d^d x \left|\psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right)\right| = \int_\Omega d^d x \int_Y d^d y \, |\psi(\boldsymbol{x}, \boldsymbol{y})| \tag{2.1}$$

Let $C_p(Y)$ the space of Y-periodic continuous functions and let us denote by $\mathbb{L}^1(\Omega; C_p(Y))$ the space of functions of the form $\psi(\boldsymbol{x}, \boldsymbol{y})$, measurable and summable in $\boldsymbol{x} \in \Omega$, with values in the Banach space of continuous functions, Y-periodic in $\boldsymbol{y}$. To $\mathbb{L}^1(\Omega; C_p(Y))$ we can associate the norm

$$\parallel \psi(\boldsymbol{x}, \boldsymbol{y}) \parallel_{\mathbb{L}^1(\Omega; C_p(Y))} := \int_\Omega d^d x \sup_{\boldsymbol{y} \in Y} |\psi(\boldsymbol{x}, \boldsymbol{y})| \tag{2.2}$$

The following proposition characterizes the elements of the $L^1(\Omega; C_p(Y))$

Proposition 2.1. *A function $\psi(\boldsymbol{x}, \boldsymbol{y})$ belongs to $\mathbb{L}^1(\Omega; C_p(Y))$ if and only if there exists a subset E (independent of $\boldsymbol{y}$) of measure zero in Ω such that*

1. *For any $\boldsymbol{x} \in \Omega / E$ the function $\boldsymbol{y} \mapsto \psi(\cdot, \boldsymbol{y})$ (i.e. ψ regarded as a function of $\boldsymbol{y}$ for $\boldsymbol{x}$ fixed) is continuous and Y-periodic.*
2. *For any $\boldsymbol{y} \in Y$ the function $\boldsymbol{x} \mapsto \psi(\boldsymbol{x}, \cdot)$ is measurable on Ω.*
3. *The function $\boldsymbol{x} \mapsto \sup_{\boldsymbol{y} \in Y} |\psi(\boldsymbol{x}, \boldsymbol{y})|$ belongs to $\mathbb{L}^1(\Omega)$:*

$$\int_\Omega d^d x \sup_{\boldsymbol{y} \in Y} |\psi(\boldsymbol{x}, \boldsymbol{y})| < \infty \tag{2.3}$$

We omit the proof of the proposition 2.1 which is sketched in [1] but we use it to derive an explicit characterization of admissible functions. Before doing that we observe that any function satisfying properties 1. and 2. is called a *Carathéodory-type function* (see appendix A).

Proposition 2.2. *Let $\psi(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{L}^1(\Omega; C_p(Y))$. Then, for any positive value of $\varepsilon > 0$, $\psi(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ is a measurable function on Ω such that*

$$\parallel \psi(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \parallel_{\mathbb{L}^1(\Omega)} \leq \parallel \psi(\boldsymbol{x}, \boldsymbol{y}) \parallel_{\mathbb{L}^1(\Omega; C_p(Y))} \tag{2.4}$$

and $\psi(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ is an "admissible" test function, i.e., satisfies (2.1).

Proof. The proof consists of three steps

Step 1.: proof of measurability

By proposition 2.1 since $\psi(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{L}^1(\Omega; C_p(Y))$ it is also a *Carathéodory-type function*. This fact entails that $\psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)$ is measurable.

Step 2.: norm upper bound

The bound (2.20) follows from the very definition of the norms.

$$\parallel \psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon) \parallel_{\mathbb{L}^1(\Omega)} := \int_\Omega d^d x \left|\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\right| \leq \int_\Omega d^d x \sup_{\boldsymbol{y}\in Y} |\psi(\boldsymbol{x},\boldsymbol{y})| \equiv \parallel \psi(\boldsymbol{x},\boldsymbol{y}) \parallel_{\mathbb{L}^1(\Omega;C_p(Y))} \tag{2.5}$$

Step 3.: admissibility

This is the most interesting for us part of the proof. For any integer n we pave the unit hypercube Y with n^d smaller hypercubes $\{Y_i\}_{i=1}^{n^d}$ each of linear size $1/n$ so that

$$Y = \bigcup_{i=1}^{n^d} Y_i \qquad \& \qquad Y_i \cap Y_j = \emptyset \quad \forall\, i \neq j \qquad \& \qquad \sum_{i=1}^{n^d} |Y_i| = 1 \tag{2.6}$$

having denoted $|Y_i|$ the volume of Y_i. On each of the Y_i we then sample a point $\boldsymbol{y}_i \in Y_i$ and define

$$\psi_n(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{n^d} \psi(\boldsymbol{x},\bar{\boldsymbol{y}}_i)\, \chi_i\left(\frac{\boldsymbol{y}}{\varepsilon}\right) \tag{2.7}$$

In (2.7) χ stands for the characteristic function of the set Y_i extended by periodicity to the full $\mathbb{R}^d$:

$$\chi_i(\boldsymbol{y}) := \begin{cases} 1 & \text{if } \boldsymbol{y} \in Y_i \bmod Y \\ 0 & \text{if } \boldsymbol{y} \notin Y_i \bmod Y \end{cases} \tag{2.8}$$

For example if $d = 1$, $Y = [0,1]$ and $Y_i = [i/n, (i+1)/n)$, $i = 0, \dots, n-1$ we have

$$\chi_i(y) = \begin{cases} 1 & \text{if } y \in \bigcup_{l\in\mathbb{Z}} \left[\frac{i}{n}+l, \frac{i+1}{n}+l\right) \\ 0 & \text{if } y \notin \bigcup_{l\in\mathbb{Z}} \left[\frac{i}{n}+l, \frac{i+1}{n}+l\right) \end{cases} \tag{2.9}$$

We will now show that the proposition holds true if we take the limit ε tending to zero for any finite n. Using this result we will then show that as n tends to infinity the sequence of the ψ_n's converges to ψ thus proving claim. Since the χ_i's are periodic they admit a Fourier series representation

$$\chi_i(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{i:\boldsymbol{k}}\, e^{2\pi\imath \boldsymbol{k}\cdot\boldsymbol{x}} \tag{2.10a}$$

$$c_{i:\boldsymbol{k}} := \int_Y d^d x\, e^{-2\pi\imath\boldsymbol{k}\cdot\boldsymbol{x}} \chi_i(\boldsymbol{x}) \tag{2.10b}$$

Since

$$\left|\int_\Omega d^d x\, \psi(\boldsymbol{x},\bar{\boldsymbol{y}}_i)\, \chi_i\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\right| \leq \int_\Omega d^d x\, |\psi(\boldsymbol{x},\bar{\boldsymbol{y}}_i)| < \infty \tag{2.11}$$

we have then

$$\lim_{\varepsilon\downarrow 0}\int_{\Omega} d^d x\,\psi\left(\boldsymbol{x},\bar{\boldsymbol{y}}_i\right)\chi_i\left(\frac{\boldsymbol{x}}{\varepsilon}\right)=$$
$$\lim_{\varepsilon\downarrow 0}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{k}}\int_{\Omega} d^d x\,\psi\left(\boldsymbol{x},\bar{\boldsymbol{y}}_i\right)\, e^{2\,\pi\,\imath\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{\varepsilon}}=\sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{k}}\lim_{\varepsilon\downarrow 0}\int_{\Omega} d^d x\,\psi\left(\boldsymbol{x},\bar{\boldsymbol{y}}_i\right)\, e^{2\,\pi\,\imath\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{\varepsilon}} \tag{2.12}$$

The rightmost term in (2.12) vanishes as a consequence of the Riemann-Lebesgue theorem for any

$$\boldsymbol{k}\neq\boldsymbol{0} \tag{2.13}$$

so that

$$\lim_{\varepsilon\downarrow 0}\int_{\Omega} d^d x\,\psi\left(\boldsymbol{x},\bar{\boldsymbol{y}}_i\right)\chi_i\left(\frac{\boldsymbol{x}}{\varepsilon}\right)=\int_{\Omega} d^d x\,\psi\left(\boldsymbol{x},\bar{\boldsymbol{y}}_i\right)\int_{Y} d^d y\,\chi_i(\boldsymbol{y}) \tag{2.14}$$

We have therefore proved that

$$\lim_{\varepsilon\downarrow 0}\int_{\Omega} d^d x\,\psi_n\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)=\sum_{i=1}^{n^d}\int_{\Omega} d^d x\int_{Y} d^d y\,\psi\left(\boldsymbol{x},\bar{\boldsymbol{y}}_i\right)\chi_i(\boldsymbol{y})=\int_{\Omega} d^d x\int_{Y} d^d y\,\psi_n\left(\boldsymbol{x},\boldsymbol{y}\right) \tag{2.15}$$

It remains to pass to the limit n tending to infinity. Let us first prove that the strong topology of $\mathbb{L}^1(\Omega;C_p(Y))$. Define

$$\delta_n(\boldsymbol{x})=\sup_{y\in Y}\left|\psi_n\left(\boldsymbol{x},\boldsymbol{y}\right)-\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\right| \tag{2.16}$$

Since $\boldsymbol{y}\mapsto\left[\psi_n\left(\boldsymbol{x},\boldsymbol{y}\right)-\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\right]$ is almost everywhere in $\boldsymbol{x}$ picewise continuous in $\boldsymbol{y}$, we have

$$\delta_n(\boldsymbol{x})=\tilde{\delta}_n(\boldsymbol{x})=\sup_{y\in Y\cap\mathbb{Q}}\left|\psi_n\left(\boldsymbol{x},\boldsymbol{y}\right)-\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\right| \tag{2.17}$$

The set $Y\cap\mathbb{Q}$ is countable and the supremum over a countable family of measurable function is also measurable (see [1] and refs therein). The continuity of ψ in $\boldsymbol{y}$ also implies

$$\lim_{n\uparrow\infty}\delta_n(\boldsymbol{x})=0 \tag{2.18}$$

Furthermore, the inequality

$$\delta_n(\boldsymbol{x})\leq 2\sup_{y\in Y}\left|\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\right| \tag{2.19}$$

guarantees that $\delta_n(\boldsymbol{x})\in\mathbb{L}^1(\Omega)$. Thus we can invoke the dominated convergence theorem to write

$$\lim_{n\uparrow\infty}\int_{\Omega} d^d x\,\left|\psi_n\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)-\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\right|\leq$$
$$\lim_{n\uparrow\infty}\int_{\Omega} d^d x\,\delta_n(\boldsymbol{x})=\lim_{n\uparrow\infty}\parallel\psi_n(\boldsymbol{x},\boldsymbol{y})-\psi(\boldsymbol{x},\boldsymbol{y})\parallel_{\mathbb{L}^1(\Omega;C_p(Y))}\int_{\Omega} d^d x\,\lim_{n\uparrow\infty}\delta_n(\boldsymbol{x})=0 \tag{2.20}$$

thus proving that the ψ_n strongly converge to ψ in $L^1(\Omega;C_p(Y))$-convergence. Gleaning all the above information, we are ready to estimate the difference

$$\left|\int_{\Omega} d^d x\,\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)-\int_{\boldsymbol{x}\in\Omega} d^d x\int_{Y} d^d y\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\right|\leq\left|\int_{\Omega} d^d x\,\left[\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)-\psi_n\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\right]\right|$$
$$+\left|\int_{\Omega} d^d x\,\psi_n\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)-\int_{\boldsymbol{x}\in\Omega} d^d x\int_{Y} d^d y\,\psi_n\left(\boldsymbol{x},\boldsymbol{y}\right)\right|+\left|\int_{\boldsymbol{x}\in\Omega} d^d x\int_{Y} d^d y\,\left[\psi_n\left(\boldsymbol{x},\boldsymbol{y}\right)-\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\right]\right| \tag{2.21}$$

which by the first inequality in (2.20) becomes

$$\left|\int_{\Omega} d^d x\, \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) - \int_{\boldsymbol{x}\in\Omega} d^d x \int_Y d^d y \psi\left(\boldsymbol{x}, \boldsymbol{y}\right)\right| \leq$$
$$\left|\int_{\Omega} d^d x\, \psi_n\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) - \int_{\boldsymbol{x}\in\Omega} d^d x \int_Y d^d y\, \psi_n\left(\boldsymbol{x}, \boldsymbol{y}\right)\right| + 2\, \parallel \psi_n(\boldsymbol{x}, \boldsymbol{y}) - \psi(\boldsymbol{x}, \boldsymbol{y}) \parallel_{\mathbb{L}^1(\Omega; C_p(Y))} \tag{2.22}$$

The first term on the right hand side vanishes in the limit ε tending to zero, the second in the limit $n \uparrow \infty$ thus showing that ψ is admissible as claimed. □

3 Convergence results for periodically oscillating functions, $\mathbb{L}^2$-case

Let as above Ω an open set in $\mathbb{R}^d$ and $Y = [0, 1]^d$ the unit cube in $\mathbb{R}^d$.

Proposition 3.1. *Let $\psi \in \mathbb{L}^2(\Omega; C_p(Y))$ and define $\psi^\varepsilon(\boldsymbol{x}) = \psi(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$. Then we have*

1. $\parallel \psi^\varepsilon(\boldsymbol{x}) \parallel_{\mathbb{L}^2(\Omega)} \leq \parallel \psi(\boldsymbol{x}, \boldsymbol{y}) \parallel_{\mathbb{L}^2(\Omega; C_p(Y))}$
2. $\psi^\varepsilon(x) \stackrel{\varepsilon\downarrow 0}{\rightharpoonup} \int_Y d^d y\, \psi(\boldsymbol{x}, \boldsymbol{y}) := \bar{\psi}(\boldsymbol{x}) \in \mathbb{L}^2(\Omega)$ *i.e. weakly in* $\mathbb{L}^2(\Omega)$
3. $\lim_{\varepsilon\downarrow 0} \parallel \psi^\varepsilon(\boldsymbol{x}) \parallel_{\mathbb{L}^2(\Omega)} \geq \parallel \psi(\boldsymbol{x}, \boldsymbol{y}) \parallel_{\mathbb{L}^2(\Omega\times Y)} \geq \parallel \bar{\psi}(\boldsymbol{x}) \parallel_{\mathbb{L}^2(\Omega)}$

Proof. 1. By definition we have

$$\parallel \psi^\varepsilon(\boldsymbol{x}) \parallel_{\mathbb{L}^2(\Omega)} = \int_\Omega d^d x\, |\psi^\varepsilon|^2 \left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) \leq \int_\Omega d^d x \sup_{\boldsymbol{y}\in Y} |\psi|^2 \left(\boldsymbol{x}, \boldsymbol{y}\right) \equiv \parallel \psi(\boldsymbol{x}, \boldsymbol{y}) \parallel_{\mathbb{L}^2(\Omega; C_p(Y))} \tag{3.1}$$

2. Let $C_0(\Omega) \otimes C_p(Y)$ the space of the continuous function with *compact* support of product form

$$\psi(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} u_{\boldsymbol{n}}(\boldsymbol{x}) v_{\boldsymbol{n}}(\boldsymbol{y}) \tag{3.2}$$

General results of functional analysis (see e.g. [2] ch. VII) guarantee that $C_0(\Omega) \otimes C_p(Y)$ is dense over $C_0(\Omega; C_p(Y))$ the space of continuous function with *compact* support. On its turn $C_0(\Omega; C_p(Y))$ is dense over $\mathbb{L}^2(\Omega; C_p(Y))$. This means that it is sufficient to prove the claim for

$$\psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) = u(\boldsymbol{x}) v\left(\frac{\boldsymbol{x}}{\varepsilon}\right) \tag{3.3}$$

As by hypothesis v is Y-periodic, it admits a Fourier series representation

$$\psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} u(\boldsymbol{x}) v_{\boldsymbol{n}}\, e^{2\pi\imath \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{\varepsilon}} \tag{3.4a}$$

$$v_{\boldsymbol{n}} = \int_Y d^d y\, e^{-2\pi\imath \frac{\boldsymbol{n}\cdot\boldsymbol{y}}{\varepsilon}} v(\boldsymbol{y}) \tag{3.4b}$$

We can therefore write

$$\psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} u(\boldsymbol{x}) v_{\boldsymbol{n}}\, e^{2\pi\imath \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{\varepsilon}} \tag{3.5}$$

For any test function $f \in \mathbb{L}^2(\Omega)$ we have

$$\lim_{\varepsilon\downarrow 0}\int_\Omega d^d x\, f(\boldsymbol{x})\,\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) = \lim_{\varepsilon\downarrow 0}\sum_{\boldsymbol{n}\in\mathbb{Z}^d} v_{\boldsymbol{n}}\int d^d x\, f(\boldsymbol{x})\; u(\boldsymbol{x})\; e^{2\pi\imath\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{\varepsilon}} = v_{\boldsymbol{0}}\int d^d x\, f(\boldsymbol{x})\; u(\boldsymbol{x}) \tag{3.6}$$

In other words we have proved that

$$u(\boldsymbol{x})v\left(\frac{\boldsymbol{x}}{\varepsilon}\right) \overset{\varepsilon\downarrow 0}{\rightharpoonup} u(\boldsymbol{x})\int_Y d^d y\, v(\boldsymbol{y}) \tag{3.7}$$

The aforementioned argument density argument reduces then to the claim that any

$$\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) \in \mathbb{L}^2\left(\mathbb{R}^d; C_p(Y)\right) \tag{3.8}$$

is amenable to the form

$$\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} u_{\boldsymbol{n}}(\boldsymbol{x})\, e^{2\pi\imath\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{\varepsilon}} \tag{3.9}$$

for some $\{u_{\boldsymbol{n}}(\boldsymbol{x})\}_{\boldsymbol{n}\in\mathbb{Z}^d} \in \mathbb{L}^2(\Omega)$. We can prove the claim by applying the Riemann-Lebesgue theorem to each term of the series.

3. For any $f \in \mathbb{L}^2(\Omega; C_p(Y))$ we have

$$0 \le \int_\Omega d^d x\left[\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) - f\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\right]^2 \tag{3.10}$$

whence

$$\begin{aligned}\int_\Omega d^d x\,\psi^2\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) &\ge \int_\Omega d^d x\left[2\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) f\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) - |f|^2\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\right] \\ &\overset{\varepsilon\downarrow 0}{\rightarrow} \int_\Omega d^d x\int_Y d^d y\,\left[2\psi(\boldsymbol{x},\boldsymbol{y})\, f(\boldsymbol{x},\boldsymbol{y}) - f^2(\boldsymbol{x},\boldsymbol{y})\right]\end{aligned} \tag{3.11}$$

Owing to the arbitrainess of f we can replace it with a sequence $\{\psi_n\}_{n=0}^\infty$ converging in $\mathbb{L}^2$ to ψ, we have

$$\int_\Omega d^d x\,\psi^2\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) \ge \int_\Omega d^d x\int_Y d^d y\,\psi^2(\boldsymbol{x},\boldsymbol{y}) \tag{3.12}$$

On the other hand using $|Y| = 1$ we can use the Cauchy-Schwartz inequality to write

$$\int_Y d^d y\,\psi(\boldsymbol{x},\boldsymbol{y})\int_Y d^d z\,\psi(\boldsymbol{x},\boldsymbol{z}) \le \left[\int_Y d^d y\,\psi^2(\boldsymbol{x},\boldsymbol{y})\right]^{1/2}\left\{\int_Y d^d y\left[\int_Y d^d z\,\psi(\boldsymbol{x},\boldsymbol{z})\right]^2\right\}^{1/2} \tag{3.13}$$

whence

$$\int_Y d^d y\,\psi^2(\boldsymbol{x},\boldsymbol{y}) \ge \left[\int_Y d^d y\,\psi(\boldsymbol{x},\boldsymbol{y})\right]^2 := \bar{\psi}^2(\boldsymbol{x}) \tag{3.14}$$

and therefore

$$\int_\Omega d^d x\,\psi^2\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right) \ge \int_\Omega d^d x\int_Y d^d y\,\psi^2(\boldsymbol{x},\boldsymbol{y}) \ge \int_\Omega d^d x\,\bar{\psi}^2(\boldsymbol{x}) \tag{3.15}$$

as claimed.

□

Appendices

A Reminder of measure theory

References for measure theory could by chapter 1 of [3] or chapter 5 of [2]. In oder to define a measurable function we need the following concept

Definition A.1 (σ-algebra)**.** *A collection $\mathcal{M}$ of subsets of a set X is said to be a σ-algebra in X if M enjoys the following properties*

1. $X \in \mathcal{M}$
2. *If $A \in \mathcal{M}$ then $A^c := X/A$ (the complement of A relative to X) also belongs to M.*
3. *If $A = \cup_{n=1}^{\infty} A_n$ and $A_n \in \mathcal{M}$ for any $n = 1, 2, \ldots$ then $A \in \mathcal{M}$*

From a σ-algebra we can define a measurable space

Definition A.2 (Measurable space)**.** *If $\mathcal{M}$ is a σ-algebra in X then X is called a measurable space and the members of $\mathcal{M}$ are called the measurable sets of X.*

A measurable function is then defined as a mapping between measurable spaces

Definition A.3 (Measurable function)**.** *Let X and Y be measurable spaces, respectively endowed with σ-algebras $\mathcal{M}$ and $\mathcal{N}$. A function*

$$f : X \mapsto Y \tag{A.1}$$

is measurable if the preimage of any $B \in \mathcal{N}$ is an element of $\mathcal{M}$:

$$\forall B \in \mathcal{N} \Rightarrow f^{-1}(B) \in \mathcal{M} \tag{A.2}$$

The general definition of Carathéodory function requires the concept of topological space. To recall such concept we observe that

Definition A.4 (Topology in X)**.** *A collection of subsets $\mathcal{T}$ of a set X is said to be a topology in X if it enjoys the following three properties*

1. *The empty set $\emptyset$ belongs to $\mathcal{T}$: $\emptyset \in \mathcal{T}$*
2. *If $\{A_i\}_{i=1}^{n}$ belong to $\mathcal{T}$ for all i ($A_i \in \mathcal{T}\ \forall\, i$) then*

$$A_1 \cap A_2 \cap \cdots \cap A_n \in \mathcal{T} \tag{A.3}$$

3. *If $\{A_i\}$ is an arbitrary collection (finite, countable or uncountable) of elements of $\mathcal{T}$ then*

$$\cup_i A_i \in \mathcal{T} \tag{A.4}$$

We are thus ready to say that

Definition A.5 (Topological space)**.** *If $\mathcal{T}$ is a topology in X then X is a topological space and the elements of $\mathcal{T}$ are the* ***open sets*** *in X.*

and to give the general definition of Carathéodory function

Definition A.6 (Carathéodory function)**.** *Let T_1, T_2 be be topological spaces and M be a measurable space. We say that*

$$f\colon T_1 \times M \mapsto T_2 \tag{A.5}$$

is a Carathéodory function if

1. *$\boldsymbol{x} \mapsto f(\boldsymbol{x}, \cdot)$ is measurable for each $\boldsymbol{x} \in T_1$.*
2. *$\boldsymbol{x} \mapsto f(\cdot, \boldsymbol{x})$ is continuous for each $\boldsymbol{x} \in T_2$.*

B Riemann-Lebesgue theorem

Theorem B.1. *Let f an $\mathbb{L}^1(I)$ function over an arbitrary interval $I = [a, b] \subset \mathbb{R}$. Then for any real β we have*

$$\lim_{\alpha \to \infty} \int_I dx\, f(x)\, \cos(\alpha\, x + \beta) = 0 \tag{B.1}$$

In particular we have

$$\lim_{\alpha \to \infty} \int_I dx\, f(x)\, \cos(\alpha\, x) = 0 \tag{B.2a}$$

$$\lim_{\alpha \to \infty} \int_I dx\, f(x)\, \sin(\alpha\, x) = 0 \tag{B.2b}$$

Proof. The proofs proceeds in steps.

Step 1.: constant function and $|I| < \infty$

If

$$f(x) = f \qquad \forall\, x \in I \tag{B.3}$$

then

$$\int_I dx\, f(x)\, \cos(\alpha\, x + \beta) = f\, \frac{\sin(\alpha\, b + \beta) - \sin(\alpha\, a + \beta)}{\alpha} \tag{B.4}$$

and

$$\lim_{\alpha \to \infty} \left| \int_I dx\, f(x)\, \cos(\alpha\, x + \beta) \right| \leq \lim_{\alpha \to \infty} |f| \frac{2}{|\alpha|} = 0 \tag{B.5}$$

Step 2.: stepwise function

If $I = \bigcup_{i=1}^n I_i$ with $I_i = [a_i, a_{i+1})$, $b = a_{n+1}$ $I_i \cap I_j = \emptyset$ for any $i \neq j$ and

$$f(x) = \sum_{i=1}^{n} f_i\, \chi_i(x) \tag{B.6}$$

for χ_i the characteristic function of I_i then

$$\int_I dx\, f(x)\, \cos(\alpha\, x + \beta) = \sum_{i=1}^{n+1} f_i \frac{\sin(\alpha\, a_{i+1} + \beta) - \sin(\alpha\, a_i + \beta)}{\alpha} \tag{B.7}$$

so that

$$\lim_{\alpha\to\infty} \left| \int_I dx\, f(x)\, \cos(\alpha\, x + \beta) \right| = \lim_{\alpha\to\infty} \sum_{i=1}^{n+1} \frac{2\,|f_i|}{|\alpha|} = 0 \tag{B.8}$$

Note that the hypothesis $f \in \mathbb{L}^1(I)$ extends immediately the result to the cases $I = \mathbb{R}$ or f having a countable number of jumps ($n = \infty$). In both cases absolute integrability implies

$$\int_I dx\, |f(x)| = \sum_i |f_i| \;=\; F \;<\; \infty \tag{B.9}$$

which on its turn entails

$$\lim_{\alpha\to\infty} \left| \int_I dx\, f(x)\, \cos(\alpha\, x + \beta) \right| = \lim_{\alpha\to\infty} \frac{2\,|F|}{|\alpha|} = 0 \tag{B.10}$$

Step 3.: integrable function over $|I| \;<\; \infty$

Riemann integrability means that for any arbitrary partition $I = \bigcup_{i=1}^{n} I_i$ with $I_i = [a_i, a_{i+1})$, $b = a_{n+1}$ $I_i \cap I_j = \emptyset$ for all $i \neq j$ we can find *for any* $\varepsilon \;>\; 0$ two stepwise functions

$$f^{(j)}(x) = \sum_{i=1}^{n} f_i^{(j)}\, \chi_i(x) \qquad j = 1, 2 \tag{B.11}$$

such that

$$f^{(1)}(x) \;\leq\; f(x) \;\leq\; f^{(2)}(x) \tag{B.12a}$$

$$\int_I dx\, [f^{(2)}(x) - f^{(1)}(x)] \;\leq\; \frac{\varepsilon}{2} \tag{B.12b}$$

By the this very definition it follows that

$$\begin{aligned} \left| \int_I dx f(x)\, \cos\left(\alpha\, x + \beta\right) \right| &\leq \left| \int_I dx [f(x) - f^{(1)}]\, \cos\left(\alpha\, x + \beta\right) \right| + \left| \int_I dx f^{(1)}\, \cos\left(\alpha\, x + \beta\right) \right| \\ &\leq \int_I dx [f^{(2)}(x) - f^{(1)}(x)] + \left| \int_I dx f^{(1)}(x)\, \cos\left(\alpha\, x + \beta\right) \right| \;\leq\; \frac{\varepsilon}{2} + \left| \int_I dx f^{(1)}(x)\, \cos\left(\alpha\, x + \beta\right) \right| \end{aligned} \tag{B.13}$$

Since **Step 2.** we can choos an α sufficiently large that

$$\left| \int_I dx f^{(1)}(x)\, \cos\left(\alpha\, x + \beta\right) \right| \;<\; \frac{\varepsilon}{2} \tag{B.14}$$

the arbitrariness of ε yields the proof.

Step 4.: $f \in \mathbb{L}^1(\mathbb{R})$

In such a case we can always choose an I with $|I| < \infty$ such that

$$\left|\int_{\mathbb{R}} dx\, f(x) \cos(\alpha\, x + \beta)\right| \leq \left|\int_I dx\, f(x) \cos(\alpha\, x + \beta)\right| + \frac{\varepsilon}{2} \tag{B.15}$$

for any $\varepsilon > 0$. Upon applying **Step 3.** to the integral on the right hand side we can prove the claim. □

It is immediate to see that the claim of the Riemann-Lebesgue theorem holds true for differentiable functions. Upon integration by parts

$$\int_I dx\, f(x) \cos(\alpha\, x + \beta) = f(x) \frac{\sin(\alpha\, x + \beta)}{\alpha}\bigg|_a^b - \int_I dx\, \frac{df}{dx}(x) \frac{\sin(\alpha\, x + \beta)}{\alpha} \tag{B.16}$$

we obtain the upper bound

$$\left|\int_I dx\, f(x) \cos(\alpha\, x + \beta)\right| \leq \frac{2\,|f(a)| \vee |f(b)|}{|\alpha|} + \frac{1}{|\alpha|} \int_I dx\, \left|\frac{df}{dx}(x)\right| \tag{B.17}$$

readily vanihing for α tending to infinity.

B.1 Counter-example

The Riemann-Lebesgue holds because of the cancellations induced by the rapid oscillations of trigonmetric functions. For this reason it may not apply to functions f the integral whereof converges over $\mathbb{R}$ also because of cancellations. As an example consider

$$\int_{\mathbb{R}} dx\, \sin x^2 \cos(\alpha\, x + \beta) = \Im \int_{\mathbb{R}} dx\, \frac{e^{\imath (x^2 + \alpha\, x + \beta)} + e^{\imath (x^2 - \alpha\, x - \beta)}}{2} \tag{B.18}$$

a change of variables yields

$$\int_{\mathbb{R}} dx\, \sin x^2 \cos(\alpha\, x + \beta) = \Im \int_{\mathbb{R}} dx\, e^{\imath \left(x^2 - \frac{\alpha^2}{4}\right)} \cos\beta \tag{B.19}$$

We can perform the integral over x by encompassing the integral in a contour over the complex plane including the line

$$z = r\, e^{\imath \frac{\pi}{4}} \tag{B.20}$$

We have then

$$\int_{\mathbb{R}} dx\, \sin x^2 \cos(\alpha\, x + \beta) = 2\, \cos\beta \Im\, e^{\imath\, \frac{\pi - \alpha^2}{4}} \int_{\mathbb{R}_+} dx\, e^{-x^2} = \sqrt{\pi} \cos\beta \sin\left(\frac{\pi - \alpha^2}{4}\right) \tag{B.21}$$

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