Let X a vector space.

Def.: $\{x_j\}_{j=1}^{\infty} \subset X$ converges strongly to $x \in X$ ($x_j \to x$) if

$$\lim_{j \to \infty} \|x_j - x\| = 0$$

Def.: $\{x_j\}_{j=1}^{\infty} \subset X$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists an $N \subset \mathbb{N}$ such that $\forall\, i, j \geq N$

$$\|x_i - x_j\| < \varepsilon$$

Every convergent sequence in a normed space X is a Cauchy sequence ($\|x_i - x\| \leq \|x_i - x_j\| + \|x_j - x\|$).

Def A normed space X is said to be complete if every Cauchy sequence in X is also convergent in X

Def A Banach space X is a complete normed vector space.

Def Let X a Banach space with norm $\|\cdot\|$. The map

$$\ell : X \to \mathbb{R} \quad \text{is a BOUNDED LINEAR FUNCTIONAL}$$

if:

(I) $\ell(\alpha x + \beta y) = \alpha \ell(x) + \beta \ell(y) \quad \forall\, x, y \in X$ and $\forall\, \alpha, \beta \in \mathbb{R}$

(II) $\exists\, C > 0 : \quad |\ell(x)| \leq C \|x\| \quad \forall\, x \in X$

Def The collection of all the bounded linear functionals on a Banach space X is called the DUAL SPACE and it is denoted by X^*

Th. X^* when equipped with the norm

$$\|\ell(x)\| = \sup_{x \neq 0} \frac{|\ell(x)|}{|x|}$$

is a Banach space.

Def A Banach space X is called reflexive if the dual of its dual is isomorphic to X:

$$(X^*)^* = X$$

Weak topology:

Def: A sequence $\{x_j\}_{j=1}^{\infty}$ of elements of a Banach space X is said to converge weakly to $x \in X$

$$x_j \rightharpoonup x$$

if $\quad \ell(x_j) \to \ell(x) \quad \forall\, \ell \in X^*$

Every strongly convergent series is also weakly convergent. The converse is NOT true.

The importance of weakly convergent series stems from the following theorem.

Th.: Let X a Banach space

① Every weakly convergent sequence in X is BOUNDED

⑪ (Eberlein-Smulian) Assume that X is reflexive ($(X^*)^* = X$). Then from every bounded sequence in X we can extract a weakly convergent subsequence.

Def: Let X be Banach. A sequence $\{\ell_n\}_{n=1}^{\infty} \subset X^*$ is said to converge weak-* to $\ell \in X^*$

$$\ell_n \rightharpoonup \ell$$

if

$$\lim_{n\to\infty} \ell_n(x) = \ell(x) \quad \forall\, x \in X$$

Furthermore if $(X^*)^* = X$ then weak-* convergence coincides with weak convergence on X

Separability and its consequences

Def: A subset $X_0 \subset X$ is called DENSE if for every $\boxed{x \in X}$ there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset X_0$ that converges to x. In other words: X_0 is dense in X if its closure is X itself:

$$\overline{X_0} = X$$

Def: A Banach space is called separable if it contains a COUNTABLE DENSE SUBSET.

The compactness theorem for sequences in X^* can be stated as follows

Th: Let X a separable Banach space. Then from any bounded sequence in X^* we can extract a weak-* convergent sequence.

Every weakly convergent sequence in X^* is weak-* convergent. The converse is not true unless X is reflexive.

HILBERT SPACES

Def: An Hilbert space H is a COMPLETE inner product space.

Every Hilbert space H is a Banach space with norm on H given by:

$$(x,x) = \|x\|^2 \implies \|x\| = |(x,x)|^{1/2}$$

All elements x of H satisfy the Cauchy-Schwartz inequality: $|(x,y)| \le \|x\|\,\|y\|$ namely

$$0 \le \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = 2 - \frac{2\,(x,y)}{\|x\|\,\|y\|} \implies (x,y) \le \|x\|\,\|y\|$$

Riesz representation: we can identify the dual of H with H itself

Th. (Riesz representation): For every $l \in H^*$ there exists a UNIQUE $y \in H$ such that

$$l(x) = (x, y) \quad \forall x \in H$$

It is therefore expedient to represent the action of $l \in H^*$ on $x \in H$ in the form of a DUAL PAIRING

$$l(x) = \langle l, x \rangle_{H^*, H} = (y, x)$$

Corollary: H is REFLEXIVE: $(H^*)^* = H$

This means that by Eberlein-Smulian theorem from every bounded sequence in X we can extract a WEAKLY CONVERGENT sequence in X. For Hilbert spaces the definition of weakly convergent simplifies to

$$x_n \rightharpoonup x \iff (x_n - x, y) \xrightarrow{n \to \infty} 0 \quad \forall y \in H.$$

FUNCTION SPACES

Prop: Let Ω a subset of $\mathbb{R}^d$. We will denote by $\mathcal{C}(\bar{\Omega})$ the space of CONTINUOUS FUNCTIONS

$$f : \bar{\Omega} \to \mathbb{R}$$

$\mathcal{C}(\bar{\Omega})$ equipped with the supremum norm

$$\|f\| = \sup_{x \in \bar{\Omega}} |f(x)|$$

is a Banach space.

$\mathcal{C}^k(\bar{\Omega})$ is the space of k-times continuously differentiable functions

$\mathcal{C}^\infty(\bar{\Omega})$ is the space of SMOOTH functions

$\mathcal{C}_0^k(\Omega)$ is the space of k-times differentiable functions with COMPACT support $\Rightarrow$ vanish at infinity

$\mathcal{C}_0^\infty(\Omega)$ is the space of SMOOTH functions with COMPACT support: BUMP functions

$$f(x) = \begin{cases} \exp\left\{-\dfrac{1}{(1-|x|^2)}\right. & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

$\boxed{L^p \text{ spaces}}$

Let $f : \Omega \to \mathbb{R}$ a Lebesgue measurable function. The L^p-norm is defined as

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p\right)^{1/p} & \text{for } 1 \le p < \infty \\ \operatorname{ess\,sup}_\Omega |f| & \text{for } p = \infty \end{cases}$$

$$\operatorname{ess\,sup}_\Omega |f| = \inf_{c \in \mathbb{R}} \{ c \mid |f| \le c \text{ almost everywhere in } \Omega \}$$

We can generalize to the case $f : \Omega \to E$ with $E = \mathbb{R}^d, \mathbb{R}^{d \times d}$ using respectively the vector, operator l^d-norm

Def: $L^p(\Omega)$ ($L^p(\Omega,E)$) is the vector space of all measurable functions $f:\Omega\to\mathbb{R}$ ($f:\Omega\to E$) for which

$$\|f\|_{L^p(\Omega)}<\infty \qquad \left(\|f\|_{L^p(\Omega,E)}<\infty\right)$$

Th (Basic properties of L^p spaces)

1. The vector space $L^p(\Omega)$ equipped with the L^p-norm defined earlier is a Banach space $\forall p\in[1,\infty]$
2. $L^2(\Omega)$ is an HILBERT SPACE equipped with the scalar product
$$(u,v)_{L^2(\Omega)}=\int_\Omega dx\; u(x)v(x) \qquad \forall\, u,v\in L^2(\Omega)$$
3. $L^p(\Omega)$ is SEPARABLE for $p\in[1,\infty)$ and REFLEXIVE for $p\in(1,\infty)$. In particular $L^1(\Omega)$ is NOT reflexive and $L^\infty(\Omega)$ is neither separable nor reflexive.

Prop (Hölder inequality): let $p\in[1,\infty]$ and define $q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$

Then
$$\int_\Omega dx\,|v(x)u(x)| \le \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)} \qquad \forall\, u\in L^p(\Omega)\ \&\ \forall\, v\in L^q(\Omega)$$

Proof: Let
$$\tilde u=\frac{|u|^p}{\|u\|^p_{L^p}} \qquad \& \qquad \tilde v=\frac{|v|^q}{\|v\|^q_{L^q}}$$

$$\log\{t\tilde u+(1-t)\tilde v\}\ge t\ln\tilde u+(1-t)\ln\tilde v \qquad \text{implies}$$
$$t\tilde u+(1-t)\tilde v\ge \tilde u^t\,\tilde v^{(1-t)} \Rightarrow \frac{\tilde u}{p}+\frac{\tilde v}{q}\ge \tilde u^{1/p}\,\tilde v^{1/q}$$
$$1=\frac{1}{p}+\frac{1}{q}=\int_\Omega dx\;\frac{\tilde u}{p}+\frac{\tilde v}{q}\ge\int_\Omega dx\;\tilde u^{1/p}\,\tilde v^{1/q} \rightarrow \|u\|_{L^p}\|v\|_{L^q}\ge\int_\Omega dx\,|u\,v| \qquad \blacksquare$$

Def A sequence $\{u_n\}_{n=1}^\infty\subset L^p(\Omega)$ $p\in[1,\infty)$ is said to converge weakly to $L^p(\Omega)$
$$u_n\rightharpoonup u \quad \text{weakly-}L^p(\Omega) \quad \text{if}$$
$$\int_\Omega dx\;u_n(x)v(x)\xrightarrow{n\to\infty}\int_\Omega dx\;u(x)v(x) \qquad \forall\, v\in L^q(\Omega) \quad \text{with } q \mid \frac{1}{p}+\frac{1}{q}=1$$

In order to prove weak convergence in L^2 it suffices to prove that
$$(u_n-u,v)\to 0 \qquad \forall\, v\in C(\bar\Omega)$$
because $C(\bar\Omega)$ is dense in $L^2(\Omega)$

Obs.: The above definition is a consequence of $\left(L^p(\Omega)\right)^*=L^q(\Omega)$ for $q \mid \frac{1}{p}+\frac{1}{q}=1$

Similarly while weak convergence in L^∞ is rarely useful weak-$*$ convergence in L^∞ is very useful since $p=\infty\Rightarrow q=1$

Def: a sequence $\{u_n\}_{n=1}^\infty\subset L^\infty(\Omega)$ converges weak-$*$ in L^∞ ($u_n\overset{*}{\rightharpoonup}u$ weak-$*$ in $L^\infty(\Omega)$)

if
$$\int_\Omega dx\;u_n(x)v(x)\xrightarrow{n\to\infty}\int dx\;u(x)v(x) \qquad \forall\, v\in L^1(\Omega)$$

Since $L^1(\Omega)$ is separable the definition implies that every bounded sequence in $L^\infty(\Omega)$ has a weak-* convergent subsequence in the sense of the previous definition.

SOBOLEV SPACES

Def Let $u, v \in L^2(\Omega)$. We say that v is the WEAK derivative of u with respect to x_i if

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega v \phi \quad \forall \phi \in C_0^\infty(\Omega)$$

Def The Sobolev space $H^1(\Omega)$ comprises all the square integrable functions $u : \Omega \to \mathbb{R}$ whose first order weak derivatives exist and are square integrable

$$H^1(\Omega) = \{ u \mid u, \nabla u \in L^2(\Omega) \}$$

Prop.: The space $H^1(\Omega)$ is separable with inner product

$$(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$$

and norm $\quad \|u\|_{H^1(\Omega)} = \{ (u,u)_{L^2(\Omega)} + (\nabla u, \nabla u)_{L^2(\Omega)} \}^{1/2}$

$H^1(\Omega)$ is REFLEXIVE: any bounded sequence in $H^1(\Omega)$ contains a weakly converging subsequence

Furthermore:

Th (Rellich compactness theorem): From every bounded sequence in $H^1(\Omega)$ we can extract a subsequence that is strongly convergent in $L^2(\Omega)$

In many applications one considers elements of $H^1(\Omega)$ vanishing on $\partial\Omega$. This is the space $H_0^1(\Omega)$:

Def The Sobolev space $H_0^1(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$ norm

Th (Poincaré inequality) Let Ω be a bounded open set in $\mathbb{R}^d$. Then there is a constant C_Ω which depends only upon the size of Ω such that for every $u \in H_0^1(\Omega)$

$$\|u\|_{L^2(\Omega)} \le C_\Omega \|\nabla u\|_{L^2(\Omega)}$$

Corollary: $\|\nabla \cdot\|_{L^2(\Omega)}$ can be adopted as a metric in $H_0^1(\Omega)$

Notation: the DUAL space of $H_0^1(\Omega)$ is usually denoted by $H^{-1}(\Omega)$.

$H^{-1}(\Omega)$ is a Banach space equipped with the norm

$$\|f\|_{H^{-1}(\Omega)} = \sup_{x \in H_0^1(\Omega) \neq 0} \frac{|\langle f, x \rangle_{H^{-1}, H_0^1}|}{\|x\|_{H_0^1(\Omega)}}$$

Banach Space - Valued Spaces

It is possible to define L^p-spaces of functions varying over spaces Ω more general than $\mathbb{R}^d$, replacing the Lebesgue integral by an integral with respect to another measure on Ω. It is also possible to work with L^p-spaces of functions taking values in an arbitrary Banach space. We illustrate these ideas.

Def Let X a Banach space with norm $\|\cdot\|_X$ and let Ω denote a subset of $\mathbb{R}^d$ not necessarily bounded. The space $\mathcal{Y} := L^p(\Omega, X)$ with $p = [1, \infty]$ consists of all measurable functions $u : \Omega \to u(x) \in X$ such that $\|u(x)\|_X \in L^p(\Omega)$

Th.: Let $Y = L^p(\Omega; X)$ with X and Ω as in the above definition. Then

① Y equipped with the norm

$$\|u\|_Y = \begin{cases} \left(\int_\Omega \|u(x)\|_X^p\right)^{1/p} & \text{for } 1 \leq p < \infty \\ \operatorname{ess\,sup}_\Omega \|u(x)\|_X & \text{for } p = \infty \end{cases}$$

is a Banach space

② If X is reflexive and $p \in (1, \infty)$ then Y is also reflexive

③ If X is separable and $p \in [1, +\infty)$ then Y is also separable

SOBOLEV SPACES OF PERIODIC FUNCTIONS

$\mathbb{T}^d$ denotes the d-dimensional torus.

Def: Functions $f : \mathbb{R}^d \to \mathbb{R}$ that satisfy

$$f(x + e_i) = f(x) \quad \forall x \in \mathbb{R}^d, \ i \in \{1, \dots, d\}$$

are called 1-periodic functions.

$C^\infty_{per}(\mathbb{T}^d)$ denotes the restriction to $\mathbb{T}^d$ of elements of $C^\infty(\mathbb{R}^d)$ that are 1-periodic.

$L^p_{per}(\mathbb{T}^d)$ is defined to be the completion of $C^\infty_{per}(\mathbb{T}^d)$ with respect to the L^p-norm

$H^1_{per}(\mathbb{T}^d)$ is the analogous extension of $H^1(\Omega)$

The Poincaré inequality does NOT hold in the space H^1_{per}. It does hold, however, if we remove the constants from this space. With this in mind we define

$$H = \left\{ u \in H^1_{per}(\mathbb{T}^d) \ \middle|\ \int_{\mathbb{T}^d} u \, dy = 0 \right\}$$

There exists a constant C_p such that

$$\|u\|_{L^2(\mathbb{T}^d)} \leq C_p \|\nabla u\|_{L^2(\mathbb{T}^d)} \quad \forall u \in H$$

Hence we can use $\|u\|_H = \|\nabla u\|_{L^2(\mathbb{T}^d)}$ $u \in H$ as the norm in H

The dual of H is

$$H^* = \{ u \in (H^1_{per}(\mathbb{T}^d))^* \mid \langle u, 1 \rangle_{(H^1_{per})^*, H^1_{per}} = 0 \}$$

i.e. comprises all elements which are ORTHOGONAL to constants.

$L^2(\Omega; L^2(\mathbb{T}^d)) := L^2(\Omega \times \mathbb{T}^d)$ is an Hilbert space with inner product

$$(u, v) = \int_\Omega \int_{\mathbb{T}^d} u(x,y)\, v(x,y)\, dx\, dy$$

Th: Let $u \in L^2(\Omega; C_{per}(\mathbb{T}^d))$, $\varepsilon > 0$ and define $u^\varepsilon(x) = u(x, x/\varepsilon)$. Then

1. $u^\varepsilon \in L^2(\Omega)$ and $\|u^\varepsilon\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega; C_{per}(\mathbb{T}^d))}$
2. $u^\varepsilon(x)$ converges to $\int_{\mathbb{T}^d} u(x,y)\, dy$ WEAKLY in $L^2(\Omega)$ as $\varepsilon \downarrow 0$
3. We have
$$\|u^\varepsilon\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega \times \mathbb{T}^d)} \quad \text{as } \varepsilon \to 0$$

Th Let $p \in [1, \infty]$ and $f \in L^p_{per}(\mathbb{T}^d)$. Set

$$f^\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad \text{almost everywhere on } \mathbb{R}^d$$

Then if $p < \infty$ as $\varepsilon \to 0$:

$$f^\varepsilon \rightharpoonup \int_{\mathbb{T}^d} f(y)\, dy \quad \text{weakly in } L^p(\Omega)$$

for any bounded subset Ω in $\mathbb{R}^d$. We also have

$$f^\varepsilon \rightharpoonup \int_{\mathbb{T}^d} f(y)\, dy \quad \text{weak-* in } L^\infty(\mathbb{R}^d)$$