EVOLUTION AND THE THEORY OF GAMES

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Transcript of my notes of 15-11-2011

►*The beginning of the lecture of 15 November was a continuation on the sex-ratio evolution as an example of a 'playing the field' situation. This part of the lecture was put together with the lecture notes of 10 November and is not repeated here.*

22. In the Hawk-Dove game, two Doves share the resource. It may happen, however, that the resource is indivisible or simply not worth to divide. For example, suppose the resource is a territory. If N is the number of offspring produced by the owner of the territory, and θN ($\theta < 1$) the number by the owner of only half the territory, and n is the number in a less favorable habitat, then sharing the resource is not worthwhile if $n > \theta N$. Yet, obtaining the whole territory is worthwhile whenever $N > n$. It is under these circumstances that one would expect some kind of contest to emerge.

Such a contest without escalation (i.e., so that no-one gets injured) is called a 'display contest'. A display contest may in fact look more intimidating than an escalated fight, precisely because each contestant tries to intimidate the other by roaring, fake charges, stamping and jumping and whatever, and moreover may last longer than a real fight. In short, a display context can be costly in terms of both time and energy, and the question is how much the contestants are ready to pay?

The payoff for obtaining the resource is $R = N - n$. The cost of displaying is assumed to be an increasing function of time and is denoted by c. If two players chose to invest a maximum cost of c_1 and c_2, respectively, then the one with the higher value will win, but he does not have to pay that cost, because the length of the contest is determined by the lower value. After all, there is no point in continuing the display if your opponent has already given up. If both players happen to choose the same cost, then the contest is decided randomly. This is called the *War of Attrition*. The payoffs thus are:

	Player 1	Player 2
$c_1 > c_2$	$R - c_2$	$-c_2$
$c_1 = c_2$	$R/2 - c_2$	$R/2 - c_2$
$c_1 < c_2$	$-c_1$	$R - c_1$

Note that the choice of c is made before the contest, but that the cost is being paid only as the contests proceeds. That is why the actual cost being paid depends on the length of the game and thus on the minimum of the two choices. This is different in the so-called *Size Game* where the players invest resources in 'weaponry' such as body size, the size of antlers or horns, so that the cost is being paid before the actual contest but without 'refund' to the winner.

Also note that as the choice of c is made before the contest, there is no exchange of information about how long the opponent is going to continue, and so the value of c is not updated. For example, if one player is about to give up, but he sees that his opponent is getting tired, then in reality he may change his mind and decide not to give up yet, because victory in in sight. This does not happen in the War of Attrition.

It is clear that there is no pure strategy that is an ESS: the higher value of c always wins and can invade. Curiously enough, this even holds if the costs exceed the value of the resource! If there is an ESS it must be a mixed strategy.

Let $F : [0, \infty) \to [0, 1]$ be the cumulative distribution function of a mixed strategy, and suppose that F is everywhere continuously differentiable. Its derivative $f = F'$ is the probability density function, and

$$F(c) = \int_0^c f(\gamma) d\gamma$$

is the probability of choosing a cost less than or equal to c. The payoff to a pure strategy c against F is

$$\pi_1(c, F) = \int_0^c (R - \gamma) f(\gamma) d\gamma - c \int_c^\infty f(\gamma) d\gamma$$

If F is evolutionarily stable, then by the Bishop-Cannings theorem we have

$$\pi_1(c, F) = \pi_1(F, F)$$

Differentiation with respect to c gives

$$(R - \gamma) f(\gamma) - \int_c^\infty f(\gamma) d\gamma + c f(c) = 0$$

which we rewrite as

$$F'(c) = \frac{1}{R}\Big(1 - F(c)\Big)$$

Solving this differential equation for the initial condition $F(0) = 0$ gives

$$F(c) = 1 - e^{-c/R}$$

with corresponding probability density function

$$f(c) = \frac{1}{R}\, e^{-c/R}$$

which belongs to the so-called exponential distribution with expectation R.

Conclusion: if there is a mixed ESS with a continuous probability density function, then this must be the exponential distribution. For the expected payoff to F against itself we find:

$$\pi_1(F,F) = \pi_1(0,F) = \int_0^0 (R-\gamma)f(\gamma)d\gamma - 0 \cdot \int_0^\infty f(\gamma)d\gamma = 0$$

In other words, against an opponent who plays the strategy F, there is no expected gain from trying to obtain the resource.

Proposition. The exponential distribution with the cumulative distribution function

$$F(c) = 1 - e^{-c/R}$$

is the unique ESS of the War of Attrition.

Proof. Since the support of one ESS cannot be contained in the support of another ESS, and F has full support, it is sufficient to show that F is an ESS; uniqueness is implied automatically.

We have $\pi_1(c,F) = \pi_1(F,F)$ for all $c \geq 0$; that is how we constructed F in the first place. Hence, for any arbitrary distribution function G on $[0,\infty]$ we have

$$\pi_1(G,F) = \int_0^\infty \pi_1(c,F)dG(c) = \int_0^\infty \pi_1(F,F)dG(c)$$

$$= \pi_1(F,F)\int_0^\infty dG(c) = \pi_1(F,F)$$

so the first ESS condition fails, and we have to check the second ESS condition.

For two arbitrary distribution functions G_1 and G_2, let $H_{G_1\times G_2}$ be the cumulative probability distribution of the actual cost of a $G_1 \times G_2$-contest, i.e.,

$$\begin{aligned}
H_{G_1\times G_2}(c) &= \text{Prob}\left\{\begin{array}{l}\text{at least one of the players gives up before or at}\\ \text{the moment the cost has reached the value } c\end{array}\right\}\\
&= 1-\text{Prob}\left\{\begin{array}{l}\text{neither player gives up before or at the}\\ \text{moment the cost has reached the value } c\end{array}\right\}\\
&= 1-\Big(1-G_1(c)\Big)\Big(1-G_1(c)\Big)\\
&= G_1(c)+G_2(c)-G_1(c)G_2(c)
\end{aligned}$$

For the total payoff, $\pi_1(G_1,G_2) + \pi_1(G_2,G_1)$, we then have

$$\begin{aligned}\pi_1(G_1,G_2)+\pi_1(G_2,G_1) &= \textstyle\int_0^\infty (R-2c)dH_{G_1\times G_2}(c)\\ &= R-2\textstyle\int_0^\infty c\, d\Big(G_1(c)+G_2(c)-G_1(c)G_2(c)\Big)\end{aligned}$$

For $G_1 = G_2 = F$ we thus have

$$\pi_1(F,F)=\frac{1}{2}R-2\int_0^\infty c\, d\Big(2F(c)-F(c)^2\Big)$$

and for $G_1 = G_2 = G$ we have

$$\pi_1(G,G)=\frac{1}{2}R-2\int_0^\infty c\, d\Big(2G(c)-G(c)^2\Big)$$

and for $G_1 = F$ and $G2 = G$ we have

$$\pi_1(F,G)+\pi_1(G,F)=R-2\int_0^\infty c\, d\Big(F(c)+G(c)-F(c)G(c)\Big)$$

Consequently,

$$\begin{aligned}\pi_1(G,G)-\pi_1(F,G) &= \pi_1(G,G)-\pi_1(F,G)-\pi_1(G,F)+\pi_1(F,F)\\ &= -2\textstyle\int_0^\infty c\, d\Big(F(c)-G(c)\Big)^2 dF(c)\\ &< 0 \quad \forall G\neq F\end{aligned}$$

which means that the second ESS condition is satisfies, and so F is indeed an ESS. □