CONFORMAL FIELD THEORY, EXERCISE SET 7

1. Consider the extended Lie algebra $\hat{\mathfrak{g}} = \widehat{L\mathfrak{g}} \oplus \mathbb{C} \cdot d$ where $d = L_0$ generates the rotations on the circle, $[L_0, T_a^n] = nT_a^n$. Define a Cartan subalgebra $\hat{\mathfrak{h}}$ of $\mathfrak{g}$ to consist of $d$, the center $K$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$; concretely, in the case of $\mathfrak{g} = \mathbf{su}(n)$, we may take $\mathfrak{h}$ equal to the commutative subalgebra of diagonal matrices. The *root vectors* in $\hat{\mathfrak{g}}$ are then elements $X$ such that $[h, X] = \alpha(h)X$ for all $h \in \hat{\mathfrak{h}}$ and the *root* $\alpha$ is a linear form $\alpha : \hat{\mathfrak{h}} \to \mathbb{C}$. Show that $\hat{\mathfrak{g}}$ is a direct sum of root subspaces and compute the dimensions of the root spaces.

2. We introduce an indefinite (symmetric) inner product in $\hat{\mathfrak{h}}$ in the previous example. Let $h_i = e_{ii} - e_{i+1,i+1}$ with $i = 1, 2, \dots, n-1$. These vectors form a basis of $\mathfrak{h}$. The nonzero products are $(h_i, h_i) = 2, (h_i, h_{i+1}) = -1$ and $(K, d) = -1$. Using this nondegenerate form on $\hat{\mathfrak{h}}$ we can identify the dual $\hat{\mathfrak{h}}^*$ with $\hat{\mathfrak{h}}$. Compute the dual vector $\lambda^\vee \in \hat{\mathfrak{h}}$ for any $\lambda \in \hat{\mathfrak{h}}^*$, that is, find a vector $\lambda^\vee$ such that $\lambda(h) = (\lambda^\vee, h)$ for all $h \in \hat{\mathfrak{h}}$. In this way we can define $(\lambda, \mu) = (\lambda^\vee, \mu^\vee)$ for $\lambda, \mu \in \hat{\mathfrak{h}}^*$.

3. There is a subset of roots, the so called simple roots $\alpha_i$, such that all roots can be expressed uniquely as a linear combination of the simple roots, with only nonnegative or nonpositive integer coefficients. We can take $\alpha_i$ equal to the root corresponding to the root vector $e_{i,i+1} \in \mathfrak{g}$ (with $i = 1, 2, \dots, n-1$) and $\alpha_0 = e_{n,1} \cdot e^{ix}$ in the loop algebra. Compute the inner products of the simple roots.

4. Some of the root subspaces have dimension larger than one. Can you characterize these roots in terms of lengths of the roots?

5. Show that the conditions for the highest weights $\lambda \in \hat{\mathfrak{h}}^*$ of unitary representations of $\hat{\mathfrak{g}}$ can be expressed as $< \lambda, \alpha_i >= 0, 1, 2, \dots$ for $i = 0, 1, \dots, n-1$. Here $< \alpha, \beta >= 2(\alpha, \beta)/(\beta, \beta)$. Note that this way of writing the unitarity conditions is independent of the scaling of the inner product.