CONFORMAL FIELD THEORY, EXERCISE SET 6

1. Show that in the case of the central extension of the loop group LG for $G = SU(n)$ the necessary conditions for a highest weight in an unitary representation are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0$ and $\lambda_1 \leq k$ for level $k = 0, -1, -2, \dots$. Here the highest weight vector is characterized by $(e_{ii} - e_{nn})v_0 = \lambda_i v_0, e_{ij} v_0 = 0$ for $i < j$ and $T_a^n v_0 = 0$ for $n > 0$.

2. Let H be a complex Hilbert space with basis e_i with $i \in \mathbb{Z}$. Let $H = H_+ \oplus H_-$ the polarization to subspaces spanned by e_i with $i \geq 0$, resp. $i < 0$. Define the infinite matrices $h_n = \sum_{n\in\mathbb{Z}} e_{i+n,i}$. Compute the commutators $[h_n, h_m]$ in the central extension of the matrix Lie algebra defined by the cocycle c on page 51 in the lecture notes. Do the same for the set of matrices $\sum_i ie_{i+n,i}$.

3. Page 52 in the lecture notes: Show that Xv_0 has finite norm when X_{+-}, X_{-+} are Hilbert-Schmidt.

4. Let $H = L^2(S^1, \mathbb{C})$. Each bounded measurable function $f : S^1 \to \mathbb{C}$ defines a linear operator $T_f : H \to H$ by point wise multiplication on square-integrable functions. Formulate a necessary and sufficient condition for the Fourier coefficients of f so that the blocks $(T_f)_{+-}$ and $(T_f)_{-+}$ are Hilbert-Schmidt operators. Hint: Compute first the relevant Hilbert-Schmidt norm in the case of a single Fourier mode $f(x) = e^{inx}$.

5. Show the Hilbert-Schmidt property above for the action of smooth vector fields on functions on the circle. Actually, with a little bit more effort you could show that the same is true for all diffeomorphisms of the circle.