CONFORMAL FIELD THEORY, EXERCISE SET 3

1. Prove directly from the commutation relations that in an unitary highest weight representation of the Virasoro algebra $h \geq 0$ and $z \geq 0$.

2. Find a pair of values (z, h) for which the Verma module $M(z, h)$ is reducible.

3. Consider the case of complex fermions. The algebra is defined by two sets b_n, b_n^* of elements with anticommutation relations

$$[b_m, b_m]_+ = 0 = [b_n^*, b_m^*]_+ \text{ and } [b_n^*, b_m]_+ = \delta_{n-m}$$

for $n, m \in \mathbb{Z}$. Define

$$L_n = -\sum (k + \frac{n}{2}) : b_{n+k}^* b_k : +\alpha \delta_n$$

where the normal ordering is defined $: b_n^* b_m := -b_m b_m^*$ when $n = m > 0$ and $: b_n^* b_m := b_n^* b_m$ otherwise. The vacuum vector is defined by $b_n v_0 = 0$ for $n < 0$ and $b_n^* v_0 = 0$ for $n \geq 0$. Show that the L_n's define a Virasoro algebra and compute the central charge z.

4. Do the exercise on the page 34 in the lecture notes.

5. Define in four space dimensions, for a $SU(n)$ valued function g, the current

$$J^i = \sum \epsilon^{ijkm} \mathrm{tr}\,(g^{-1}\partial_j g)(g^{-1}\partial_k g)(g^{-1}\partial_m g),$$

where $\epsilon^{1234} = +1$ and ϵ is totally antisymmetric, and show that $\sum \partial_i J^i = 0$. Why is this relevant for the homotopy invariance of $\Gamma(g)$, when g is defined on a closed 3-manifold?