CONFORMAL FIELD THEORY, EXERCISE SET 2

1. The group $G = SL(2,\mathbb{R})$ acts in the complex upper half plane $\mathbb{C}_+$. Show that $\mathbb{C}_+ = G/H$ and find the subgroup H.

2. The Lie algebra of $O(n)$ for $n \geq 3$ does not have any nontrivial central extensions. Show that the Lie algebra of the semidirect sum $\mathfrak{o}(n) \oplus_\rho \mathbb{R}^{2n}$ has a nontrivial central extension. Here the semidirect sum is defined using the automorphisms $\rho(A)(x \oplus y) = Ax \oplus Ay$ of $\mathbb{R}^{2n}$, where $A \in \mathfrak{o}(n)$ and $x, y \in \mathbb{R}^n, x \oplus y \in \mathbb{R}^{2n}$ [Compare with Exercise 6/1].

3. Show that $SL(2,\mathbb{R})$ is a subgroup of $\mathrm{Diff}(S^1)$. What is the corresponding subalgebra of the Witt algebra?

4. In the case of $\mathbb{R}^{1,1}$, following the general discussion on conformal groups in $\mathbb{R}^{p,q}$, one can define the *finite-dimensional* conformal group as $SO_0(2,2)$. This has dimension $= 6$. Using the light cone coordinates $x_\pm = x \pm y$ show that the group acts as fractional linear transformations [compare with Exercise 1] and thus $SO_0(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

5. A semidirect product of groups G, A when A is abelian is defined by the multiplication $(g,a)(g',a') = (gg', \rho(g)a' + a)$ where ρ is a homomorphism from G to the group $\mathrm{Aut}(A)$ of automorphisms of A. In the case of Exercise 2 construct the Lie group corresponding to the Lie algebra, both in the nonextended and the centrally extended cases.

6. The Virasoro algebra is a central extension of the Witt algebra. Likewise, the group $\mathrm{Diff}(S^1)$ has a central extension by the abelian group $\mathbb{R}$. The central extension is defined by a *group cocycle* $\Omega : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \to \mathbb{R}$. What are the conditions on Ω which are needed for the group axioms in the extension? Show that the *Bott cocycle*

$$B(F,G) = \int_0^{2\pi} \ln|F'(G(t))| d \ln|G'(t)|,$$

where the diffeomorphims are written as mod 2π periodic maps $F, G : [0, 2\pi] \to \mathbb{R}$, corresponds to the Virasoro cocycle.