CONFORMAL FIELD THEORY, EXERCISE SET 11

1. Consider the charged fermion field $\psi^{\pm} = \sum \psi_n^{\pm} z^{-n-1}$ with $[\psi^+(z), \psi^-(w)]_+ = \delta(z-w)$. Prove the operator product expansion

$$Y(v^\lambda, z)\psi^+(w) \sim \frac{\partial\psi^+(w)}{z-w} + \lambda\frac{\psi^+(w)}{(z-w)^2}$$
$$Y(v^\lambda, z)\psi^-(w) \sim \frac{\partial\psi^-(w)}{z-w} + (1-\lambda)\frac{\psi^-(w)}{(z-w)^2}$$

for the vector $v^\lambda = (1-\lambda)\psi^+_{-2}\psi^-_{-1}|0> + \lambda\psi^-_{-2}\psi^+_{-1}|0>$.

2. Show that $Y(v^\lambda, z)$ is a Virasoro field with central charge $c_\lambda = -12\lambda^2 + 12\lambda - 2$.

3. Define the bosonic field $\alpha(z) =: \psi^+(z)\psi^-(z) :$. Show that $[\alpha(z), \alpha(w)] = \partial_w\delta(z-w)$.

4. Compare the commutators $[: \alpha(z)\alpha(z) : , \psi^{\pm}(w)]$ and $[W(z), \psi^{\pm}(w)] \equiv [: \partial\psi^+(z)\psi^-(z) : + : \partial\psi^-(z)\psi^+(z) : , \psi^{\pm}(w)]$ and use this to show that $: \alpha(z)\alpha(z) : = W(z)$.

5. Define the operators $E_{ij} = \psi^+_{-i}\psi^-_{j-1}$ for all $i, j \in \mathbb{Z}$ and compute $E_{ij}|m>$ for $i \le j$ when $|m> = \psi^+_{-m} \dots \psi^+_{-1}|0>$ for $m \ge 0$ and $|m> = \psi^-_m \dots \psi^-_{-1}|0>$ for $m < 0$.