CONFORMAL FIELD THEORY, EXERCISE SET 10

1. Show that a holomorphic vertex operator $Y(a,z) = \sum_{n<0} a_{(n)} z^{-n-1}$ defines a commutative and associative algebra by $a \cdot b = a_{(-1)} b$ for $a, b \in V$.

2. Show that a holomorphic vertex operator can be written as $Y(a,z) = e^{zT} a$ where $a \in V$, and V is an associative algebra with a unit and a derivation $T : V \to V$, $T(ab) = T(a)b + aT(b)$.

3. Let a_n be the generators of the Heisenberg algebra $[a_n, a_m] = n\delta_{n+m}$ acting in the Fock space of polynomials in $x_1, x_2, \dots$ in the usual way, $a_n = \frac{\partial}{\partial x_n}$ for $n > 0$ and $a_n = -nx_{-n}$ for $n < 0$ and $a_0 = 0$. Compute the commutation relations for the vertex operator

$$Y(\frac{1}{2}x_1^2 + \lambda x_2, z) = \frac{1}{2} \sum_{k \neq 0} \left(\sum_{n+m=k} a_n a_m - \lambda(k+1) a_k \right) z^{-k-2} + \sum_{n>0} a_{-n} a_n z^{-2}.$$

4. Derive the operator product expansion for the *complex* fermionic field $\psi(z) = \sum_{n \in \mathbb{Z}+1/2} b_n z^{-n-\frac{1}{2}}$ and $\psi^*(z) = \sum_{n \in \mathbb{Z}+1/2} b_n^* z^{n-1/2}$ with $b_n b_m^* + b_m^* b_n = \delta_{n-m}$ (compare with exercise 1/9). Define the translation operator T so that this field defines a vertex operator, according to the existence Theorem 10.24 in Schottenloher's book.

5. Show that the Virasoro field in the exercise 4 can be written as

$$L(z) = \frac{1}{2} : (\partial\psi^*(z))\psi(z) : -\frac{1}{2} : \psi^*(z)\partial\psi(z) : .$$