CONFORMAL FIELD THEORY, EXERCISE SET 1

1. The metric $< x, y >= x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$ in $\mathbb{R}^{1,3}$ induces by restriction a (pseudo) metric on tangent vectors to the surface $(x^0)^2 - \cdots - (x^3)^2 = -1$. What is the signature of this metric?

2. Consider a special conformal transformation K_b in $N^{p,q}$ and in $\mathbb{R}^{p,q}$. The set of points $x \in \mathbb{R}^{p,q}$ with $1 - 2 < x, b > +x^2b^2 = 0$ is a singular in $\mathbb{R}^{p,q}$ with respect to K_b. How are the corresponding points transformed in $N^{p,q}$, i.e., where are the points $K_b(i(x))$ lying?

3. Let J be a nonsingular ($\det J \neq 0$) real *antisymmetric* $2n \times 2n$ matrix and define the bilinear form $(x, y) = x^tJy$ in $\mathbb{R}^{2n}$. The real *symplectic group* $Sp(2n)$ consists of matrices A such that $(Ax, Ay) = (x, y)$ for all $x, y \in \mathbb{R}^{2n}$. Show that $Sp(2n)$ is indeed a Lie group. Compute its Lie algebra, following the Example 2 on page 10 in the lecture notes.

4. Show that $\dim Sp(2n) = n(2n + 1)$.

5. Let X, Y be $n \times n$ matrices. Show that

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\ldots}$$

where the dots denote higher order terms in X, Y. This is the Baker-Campbell-Hausdorff formula. [See a Wiki article for more details!]

6. Let $\mathfrak{g}$ and $\mathfrak{a}$ be a Lie algebras and $\rho : \mathfrak{g} \to \mathrm{Der}\,\mathfrak{a}$ a linear map to the algebra of *derivations* of $\mathfrak{a}$, i.e.,

$$\rho(X)([A, B]) = [\rho(X)A, B] + [A, \rho(X)B].$$

Assume further that $\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$ for all X, Y. Show that the bracket

$$[(X, A), (Y, B)] = ([X, Y], \rho(X)B - \rho(Y)B + [A, B])$$

defines a Lie algebra structure in the direct sum $\mathfrak{g} \oplus \mathfrak{a}$ of vector spaces.