Vektorianalyysi, harj. 8

8.1. Olk. $f: R^3 \to R^3$, $f(x,y,z) = (x+y, xy, z^3)$
Selvästi $f \in C^1(R^3)$

$$\Rightarrow \det f'(x,y,z) = \begin{vmatrix} 1 & 1 & 0 \\ y & x & 0 \\ 0 & 0 & 3z^2 \end{vmatrix} = 3z^2(x-y) = 0$$

$\Leftrightarrow \quad x = y$ tai $z = 0$

Käänteiskuvauslauseen* mukaan
f on lokaali diffeomorfismi
(x,y,z):n ympäristössä
joss $\det f'(x,y,z) \neq 0$,
siis joss
$x \neq y$ & $z \neq 0$.

* kts. luennot, L 7.14.3.

T.2. Merk. ellipsoidia

$$E = \{(x,y,z) \in \mathbb{R}^3 ; 2x^2+2y^2+z^2=6\}$$

Kun $z > 0$, $(x,y,z) \in E$

$$\iff \quad z := f(x,y) = (6-2x^2-2y^2)^{\frac{1}{2}}$$

$$\implies \quad \partial_1 f(x,y) = -2x\,(6-2x^2-2y^2)^{-1/2}$$

$$= -2x / f(x,y)$$

&

$$\partial_2 f(x,y) = -2y / f(x,y)$$

$$r(x,y) := (x,y,f(x,y))$$

$$f(1,1) = (6-2\cdot 1^2 - 2\cdot 1^2)^{\frac{1}{2}} = \sqrt{2},$$

(ei niin, 2, joka lie painovirhe)

$$\partial_1 r(x,y) = \left(1, 0, \frac{-2x}{f(x,y)}\right)$$

$$\& \quad \partial_2 r(x,y) = \left(0, 1, \frac{-2y}{f(x,y)}\right)$$

$\implies$ Tangenttitason normaali:

$$\bar{n} = \partial_1 r(1,1) \times \partial_2 r(1,1)$$

$$= \left(1, 0, -\frac{2}{\sqrt{2}}\right) \times \left(0, 1, -\frac{2}{\sqrt{2}}\right)$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -\sqrt{2} \\ 0 & 1 & -\sqrt{2} \end{vmatrix} = (-\sqrt{2}, \sqrt{2}, 1)$$

& taso:

$$((x,y,z) - (1,1,\sqrt{2}))\cdot(-\sqrt{2}, \sqrt{2}, 1) = 0$$

$$\iff -\sqrt{2}\,x + \sqrt{2}\,y + z = \sqrt{2}$$

8.3. Merk. $f(x,y,z) = x^2 - z e^{x+y+z}$

$$\partial_3 f(x,y,z) = -e^{x+y+z} - z e^{x+y+z}$$

$$\Rightarrow \partial_3 f(0,0,0) = -e^0 - 0e^0 = -1 \neq 0$$

Implisiittifunktiolause $\Rightarrow \exists$ $(0,0)$:n ympäristö $D \subset \mathbb{R}^2$ & $\varphi \in C^1(D)$ s.e. kun $(x,y) \in D$

$$f(x,y,z) = 0 \Leftrightarrow z = \varphi(x,y)$$

Nyt $f(0,0,z) = -z e^z = 0 \Leftrightarrow z = 0$
joten $\varphi(0,0) = 0$.
Kun $(x,y) \in D$,
$f(x,y,\varphi(x,y)) = 0$

$$\Rightarrow 0 = \partial_1 f(x,y,\varphi(x,y)) + \partial_3 f(x,y,\varphi(x,y)) \cdot \partial_1 \varphi(x,y)$$

$$= 2x - z e^{x+y+z} + (-e^{x+y+z} - z e^{x+y+z}) \partial_1 \varphi(x,y)$$

Sij. $x = y = z = \varphi(x,y) = 0$

$$\Rightarrow 0 = 0 - 1 \cdot \partial_1 \varphi(0,0) \Rightarrow \partial_1 \varphi(0,0) = 0$$

&

$$0 = \partial_2 f(x,y,\varphi(x,y)) + \partial_3 f(x,y,\varphi(x,y)) \cdot \partial_2 \varphi(x,y)$$

$$= 0 + (-1 - z) e^{x+y+z} \partial_2 \varphi(x,y)$$

Sij. $x = y = z = \varphi(x,y) = 0$

$$\Rightarrow \partial_2 \varphi(0,0) = 0$$

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Merk. $r(x,y) = (x, y, \varphi(x,y))$

$$\Rightarrow \quad \partial_1 r(x,y) = (1, 0, \partial_1 \varphi(x,y)),$$

$$\partial_2 r(x,y) = (0, 1, \partial_2 \varphi(x,y))$$

$\Rightarrow$ normaali:

$$\bar{n} = \partial_1 r(0,0) \times \partial_2 r(0,0)$$

$$= (1,0,0) \times (0,1,0) = (0,0,1)$$

$\bar{0}$:n kautta kulkeva tangenttitaso:

$$((x', y', z') - (0,0,0)) \cdot \bar{n} = 0$$

$$\Leftrightarrow (x', y', z') \cdot (0,0,1) = 0$$

$$\Leftrightarrow \quad \underline{z' = 0}$$

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8.4. $f(x,y)=0 \Leftrightarrow f(x,\varphi(x))=0$

$\varphi(x_0)=y_0$

$$\Rightarrow 0 = \frac{df}{dx}(x,\varphi(x)) = \frac{\partial f}{\partial x}(x,\varphi(x)) + \frac{\partial f}{\partial y}(x,\varphi(x))\,\varphi'(x).$$

Khi $x=x_0$, thì $\varphi(x)=\varphi(x_0)=y_0$

$$\Rightarrow 0 = \frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\,\varphi'(x_0)$$

$$\Leftrightarrow \varphi'(x_0) = -\frac{\partial_x f(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)}$$

$$\Rightarrow 0 = \frac{d^2 f}{dx}\, f(x,\varphi(x))$$

$$= \frac{\partial^2}{\partial x^2} f(x,\varphi(x)) + \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,\varphi(x))\,\varphi'(x)$$

$$+ \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,\varphi(x))\,\varphi'(x)$$

$$+ \frac{\partial^2}{\partial y^2} f(x,\varphi(x))\,\varphi'(x)^2$$

$$+ \frac{\partial f}{\partial y}(x,\varphi(x))\,\varphi''(x)$$

$$\Rightarrow \varphi''(x_0) = -\frac{1}{\frac{\partial}{\partial y} f(x_0,\varphi(x_0))}\left\{\frac{\partial^2}{\partial x^2} f(x_0,\varphi(x_0))\right.$$

$$+ \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x_0,\varphi(x_0))\,\varphi'(x_0)$$

$$+ \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x_0,\varphi(x_0))\,\varphi'(x_0)$$

$$\left. + \frac{\partial^2}{\partial y^2} f(x_0\,\varphi(x_0))\,\varphi'(x_0)^2\right\}$$

$$= -\frac{1}{\partial_y f(x_0, y_0)}\left\{ \partial_x^2 f(x_0, y_0) \right.$$

$$-\frac{\partial_x f(x_0,y_0)}{\partial_y f(x_0,y_0)}\left[\partial_y \partial_x f(x_0, y_0) + \partial_x \partial_y f(x_0, y_0)\right]$$

$$\left. + \partial_y f(x_0, y_0)\left(\frac{\partial_x f(x_0, y_0)}{\partial_y f(x_0, y_0)}\right)^2 \right\}$$

8.5. $f(x,y) = x^3 - xy^2$

$A := \{(x,y) \in \mathbb{R}^2 ; \; x^2+y^2 \leq 1\}$

1° Mahdolliset ääriarvot A:n sisäpisteissä ($x^2+y^2<1$):

$\nabla f(x,y) = (3x^2-y^2, -2xy) = (0,0)$

$\Leftrightarrow \begin{cases} 3x^2-y^2=0 \quad (*) \\ xy=0 \Leftrightarrow x=0 \text{ t. } y=0 \end{cases}$

Sij. $(*)$:een

$\Rightarrow 3x^2-0^2=0$ t. $0-y^2=0$

$\Rightarrow x=y=0$

$\Rightarrow$ Ainoa kriittinen piste & siis ainoa mahdollinen ääriarvokohta A:n sisällä on $(0,0)$ &

$f(0,0)=0$

2° A:n reuna ∂A: Merk. $g(x,y)=x^2+y^2-1$
(Siis $(x,y)\in\partial A$ joss $g(x,y)=0$.)

$\nabla f(x,y) = \lambda \nabla g(x,y)$ & $g(x,y)=0$

$\Leftrightarrow (3x^2-y^2, -2xy) = \lambda(2x, 2y)$ & $g(x,y)=0$

$\Leftrightarrow \begin{cases} 3x^2-y^2 = 2\lambda x \\ -2xy = 2\lambda y \Leftrightarrow -2x=2\lambda \text{ t. } y=0 \\ x^2+y^2=1 \end{cases}$

$\Leftrightarrow x=-\lambda$ t. $y=0$

i) $y=0$: $x^2+0^2=1 \Rightarrow x=\pm 1$

$\Rightarrow 3(\pm1)^2-0^2 = \pm 2\lambda \Leftrightarrow \lambda = \pm\frac{3}{2}$

$f(\pm1,0) = (\pm1)^3 - (\pm1)\cdot 0 = \pm 1$

ii) $\lambda = -x$:

$$\Rightarrow \begin{cases} 3x^2 - y^2 = 2(-x)\cdot x = -2x^2 \\ x^2 + y^2 = 1 \\ (\cancel{x}\ -2xy = -2xy) \end{cases}$$

$$1 \Leftrightarrow \begin{cases} 5x^2 = y^2 \\ 1 - x^2 = y^2 \end{cases} \Rightarrow 5x^2 = 1 - x^2 \Leftrightarrow x^2 = \frac{1}{6} \Leftrightarrow x = \pm \frac{1}{\sqrt{6}}$$

$$\Rightarrow y^2 = 1 - \left(\frac{\pm 1}{\sqrt{6}}\right)^6 = \frac{5}{6} \Rightarrow y = \pm\sqrt{5/6}$$

$$\& \quad f\left(\pm\sqrt{\tfrac{1}{6}}, \pm\sqrt{\tfrac{5}{6}}\right) = \pm\left(\frac{1}{6}\right)^{3/2} - \left(\pm\sqrt{\tfrac{1}{6}}\right)\left(\pm\sqrt{\tfrac{5}{6}}\right)^2$$

$$= \pm\left[\left(\frac{1}{6}\right)^{\frac{3}{2}} - \frac{5}{6^{3/2}}\right] = \pm\left(-\frac{4}{6^{3/2}}\right)$$

$$= \mp \frac{4}{2^{3/2}} \cdot \frac{1}{3^{3/2}} = \mp \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\right)^3$$

("−", kun $x = +\frac{1}{\sqrt{6}}$)

$1^\circ - 2^\circ \Rightarrow$

$$-1 = f(-1,0) < \left\{ \begin{matrix} f(0,0) \\ f\left(\pm\sqrt{\tfrac{1}{6}}, \pm\sqrt{\tfrac{5}{6}}\right) \end{matrix} \right\} < f(1,0)$$

$$\Rightarrow \min_{(x,y)\in A} f(x,y) = -1 \quad \&$$

$$\max_{(x,y)\in A} f(x,y) = +1.$$

II RATK: ... Reunalla $y^2 = 1 - x^2$ & $\varphi(x) := f(x, \pm\sqrt{1-x^2})$ $= x^3 - x(1-x^2) = 2x^3 - x$, $-1 \le x \le 1$ & $\varphi(-1) = -1$, $\varphi(1) = 1$, $\varphi'(x) = 6x^2 - 1 = 0$ $\Leftrightarrow x = \pm 1/\sqrt{6}$ & $\varphi(\pm 1/\sqrt{6}) = \ldots$ kuten ii):ssa.

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8.6. $f(x,y) = x^3 + y^3 - 3xy$

$A := \{(x,y) ; 0 \le x \le 1, 0 \le y \le 2\}$

1° $\nabla f(x,y) = (3x^2 - 3y, 3y^2 - 3x) = (0,0)$

$\Leftrightarrow \begin{cases} x^2 - y = 0 \\ y^2 - x = 0 \end{cases} \Rightarrow x^4 = y^2 = x$

$\Leftrightarrow x = 0$ t. $x^3 = 1$

$\Leftrightarrow x = 0$ t. $x = 1$

$x = 0 \Rightarrow y = 0^2 = 0$
$x = 1 \Rightarrow y = 1^2 = 1$

$\Rightarrow$ Kriittiset pisteet $(0,0), (1,1) \in \partial A$.
$\Rightarrow$ Ei ääriarvokohtia A:n sisällä.

2° $(x,y) \in \partial A$:

i) $y = 0$, $0 \le x \le 1$: $g_1(x) := f(x,0) = x^3$

Selvästi $\min_{0 \le x \le 1} g_1(x) = \underline{0}$ & $\max_{0 \le x \le} g_1(x) = \underline{1}$

ii) $y = 2$, $0 \le x \le 1$: $g_2(x) := f(x,2) = x^3 + 8 - 6x$

$g_2(0) = 8$, $g_2(1) = 3$, $g_2'(x) = 3x^2 - 6 = 0$
$\Leftrightarrow x = \pm\sqrt{2} \notin [0,1]$

$\Rightarrow \max_{0 \le x \le 1} g_2(x) = \underline{8}$ & $\min_{0 \le x \le 1} g_2(x) = \underline{3}$

iii) $x = 0$, $0 \le y \le 2$: $g_3(y) := f(0,y) = y^3$

Selvästi $\min_{0 \le y \le 2} g_3(y) = \underline{0}$, $\max_{0 \le y \le 2} g_3(y) = \underline{1}$

iv) $x = 1,\ 0 \le y \le 2: g_4(y) = f(1,y)$
$= y^3 - 3y + 1$

$g_4(0) = 1,\ g_4(2) = 8 - 6 + 1$
$= 3$

$g_4'(y) = 3y^2 - 3 = 0 \Rightarrow y = (\pm)1$ & $g_4(1) = -1$

1° & 2° i)–iv)

$$\min_{(x,y)\in A} f(x,y) = \min\{-1, 0, 3, 1\}$$
$$= -1 = f(1,1)$$

&

$$\max_{(x,y)\in A} f(x,y) = \max\{0, 1, 8, 3\}$$
$$= 8 = g_2(0) = f(0,2)$$