Malliavin calculus, Problem sheet n.4 (30.03.11), spring 2011.

We define the Ornstein-Uhlenbeck process as a stationary gaussian Markov process $(X_t(\omega) : t \in \mathbb{R})$ such that if $F(x,y)$ is a bounded measurable function

$$E(F(X_s, X_t)) = E(F(X_0, X_{|t-s|}) = E\big(F(Z, Z\exp(-\lambda|t-s|) + Y\sqrt{1-e^{-2\lambda|t-s|}}\big)$$

where Z and Y are two independent standard gaussian random variables, with 0 mean and variance 1, and $\lambda > 0$.

Kolmogorov continuity criterium says that a one dimensional process $(X_t : t \in \mathbb{R})$ with values in $\mathbb{R}$ can be constructed on the canonical space of continuous functions (in other words admits a version with continous sample paths) when the covariance satisfies the following criterium:

There are constants $\alpha, C, \beta > 0$ such that for all $t, s \in \mathbb{R}$

$$E(|X_t - X_s|^\alpha) \le C|t-s|^{1+\beta}$$

Exercise 1) Compute the covariance $E_P(X_s, X_t)$.

Exercise 2) Check that X_t and X_0 have the same law.

Exercise 3) Use Kolmogorov continuity criterium to show that the OU-process admits a version with continuous sample paths.

Hint: $(1 - \exp(-x)) \le x$

Exercise 4) Consider the trajectories of the process on $\mathbb{R}$ as random variables in the Banach space

$$E = C(\mathbb{R}) = \{ \text{ bounded continuous functions } f : \mathbb{R} \to \mathbb{R}\}$$

with the supremum norm $|\cdot|_\infty$.

Show that the Cameron Martin space of the OU process coincides with the Cameron Martin space of Brownian motion considered in the lectures, but with a different scalar product.

The dual space E^* is the space of Radon measures, that is signed measures which are finite on the compact subsets of $\mathbb{R}$. with the duality

$$\langle \mu, f\rangle = \int_{\mathbb{R}} f(x)\mu(dx)$$

If you want you can work instead on a finite interval $[0,T]$, $E = C([0,T])$ and E^* is the space of finite signed measures on $[0,T]$

Hint: apply the Kernel operator to a measure $\mu \in E$,

$$(K\mu)(t) = \int_E X(t)\Big(\int_0^T X(s)\mu(ds)\Big)\Gamma(dX)$$

where $\Gamma(dX)$ is the gaussian measure on the Banach space E. Use Fubini. Then compute the scalar product

$$\langle h, f\rangle_H = \langle \nu, K\mu\rangle$$

for $h = K\nu$,$f = K\mu$ with $\nu, \mu \in E^*$, and complete the image space $K(E^*)$ with respect to this scalar product.

Hint: In the calculations remember that the derivative of a step function in the sense oif distributions is a Dirac delta function (point mass).