0 On tensor products of vector spaces

A crucial concept in the course is that of a tensor product of vector spaces. Here, vector spaces can be over any field $\mathbb{K}$.

Definition 1. *Let V_1, V_2, W be vector spaces. A map $\beta : V_1 \times V_2 \to W$ is called* <u>bilinear</u> *if for all $v_1 \in V_1$ the map $v_2 \mapsto \beta(v_1, v_2)$ is linear $V_2 \to W$ and for all $v_2 \in V_2$ the map $v_1 \mapsto \beta(v_1, v_2)$ is linear $V_1 \to W$.*

Multilinear maps $V_1 \times V_2 \times \cdots \times V_n \to W$ are defined similarly.

The tensor product is a space which allows us to replace some bilinear (more generally multilinear) maps by linear maps.

Let V_1 and V_2 be two vector spaces. A <u>tensor product</u> of V_1 and V_2 is a vector space U together with a bilinear map $\phi : V_1 \times V_2 \to U$ such that the following universal property holds: for any bilinear map $\beta : V_1 \times V_2 \to W$, there exists a unique linear map $\bar{\beta} : U \to W$ such that the diagram

$$\begin{tikzcd} V_1 \times V_2 \arrow[rr, "\beta"] \arrow[dr, "\varnothing"'] & & W \\ & U \arrow[ur, dashed, "\bar{\beta}"'] & \end{tikzcd} ,$$

commutes.

Proving the uniqueness (up to canonical isomorphism) of an object defined by a universal isomorphism is a standard exercise in abstract nonsense. Indeed, if we suppose U' with a bilinear map $\phi' : V_1 \times V_2 \to U'$ is another tensor product, then the universal property of U gives a linear map $\bar{\phi}' : U \to U'$ such that $\phi' = \bar{\phi}' \circ \phi$. Likewise, the universal property of U' gives a linear map $\bar{\phi} : U' \to U$ such that $\phi = \bar{\phi} \circ \phi'$. Combining these we get

$$\mathrm{id}_U \circ \phi \;=\; \phi \;=\; \bar{\phi} \circ \phi' \;=\; \bar{\phi} \circ \bar{\phi}' \circ \phi.$$

But here are two ways of factorizing the map ϕ itself, so by the uniqueness requirement in the universal property we must have equality $\mathrm{id}_U = \bar{\phi} \circ \bar{\phi}'$. By a similar argument we get $\mathrm{id}_{U'} = \bar{\phi}' \circ \bar{\phi}$. We conclude that $\bar{\phi}$ and $\bar{\phi}'$ are isomorphisms (and inverses of each other).

Now that we know that tensor product is unique (up to canonical isomorphism), we use the following notations

$$U \;=\; V_1 \otimes V_2 \qquad \text{and}$$
$$V_1 \times V_2 \;\ni\; (v_1, v_2) \;\overset{\phi}{\mapsto}\; v_1 \otimes v_2 \;\in\; V_1 \otimes V_2.$$

An explicit construction which shows that tensor products exist is left as an exercise in *Problem sheet 2*. The same exercise establishes two fundamental properties of the tensor product:

- If $(v_i^{(1)})_{i \in I}$ is a linearly independent collection in V_1 and $(v_j^{(2)})_{j \in J}$ is a linearly independent collection in V_2, then the collection $\left(v_i^{(1)} \otimes v_j^{(2)}\right)_{(i,j) \in I \times J}$ is linearly independent in $V_1 \otimes V_2$.
- If the collection $(v_i^{(1)})_{i \in I}$ spans V_1 and the collection $(v_j^{(2)})_{j \in J}$ spans V_2, then the collection $\left(v_i^{(1)} \otimes v_j^{(2)}\right)_{(i,j) \in I \times J}$ spans the tensor product $V_1 \otimes V_2$.

It follows that if $(v_i^{(1)})_{i \in I}$ and $(v_j^{(2)})_{j \in J}$ are bases of V_1 and V_2, respectively, then

$$\left(v_i^{(1)} \otimes v_j^{(2)}\right)_{(i,j) \in I \times J}$$

is a basis of the tensor product $V_1 \otimes V_2$. In particular if V_1 and V_2 are finite dimensional, then

$$\dim (V_1 \otimes V_2) \;=\; \dim (V_1) \dim (V_2).$$

A tensor of the form $v^{(1)} \otimes v^{(2)}$ is called a simple tensor. By the second property, any $t \in V_1 \otimes V_2$ can be written as a linear combination of simple tensors

$$t \;=\; \sum_{\alpha=1}^{n} v_\alpha^{(1)} \otimes v_\alpha^{(2)},$$

for some $v_\alpha^{(1)} \in V_1$ and $v_\alpha^{(2)} \in V_2$, $\alpha = 1, 2, \ldots, n$. Note, however, that such an expression is by no means unique! The smallest n for which it is possible to write t as a sum of simple tensors is called the rank of the tensor, denoted by $n = \mathrm{rank}(t)$. An obvious upper bound is $\mathrm{rank}(t) \leq \dim (V_1) \dim (V_2)$ (although this is clearly useless whenever V_1 or V_2 is infinite dimensional). The following useful observation shows that one can do much better in general.

Lemma 1. *Suppose that*

$$t \;=\; \sum_{\alpha=1}^{n} v_\alpha^{(1)} \otimes v_\alpha^{(2)},$$

where $n = \mathrm{rank}(t)$. Then both $(v_\alpha^{(1)})_{\alpha=1}^{n}$ and $(v_\alpha^{(2)})_{\alpha=1}^{n}$ are linearly independent collections.

Proof. Suppose, by contraposition, that there is a linear relation

$$\sum_{\alpha=1}^{n} c_\alpha v_\alpha^{(1)} = 0,$$

where not all the coefficients are zero. We may assume that $c_n = 1$. Thus $v_n^{(1)} = -\sum_{\alpha=1}^{n-1} c_\alpha v_\alpha^{(1)}$ and using bilinearity we simplify t as

$$t \;=\; \sum_{\alpha=1}^{n-1} v_\alpha^{(1)} \otimes v_\alpha^{(2)} + v_n^{(1)} \otimes v_n^{(2)} \;=\; \sum_{\alpha=1}^{n-1} v_\alpha^{(1)} \otimes v_\alpha^{(2)} - \sum_{\alpha=1}^{n-1} c_\alpha \, v_\alpha^{(1)} \otimes v_n^{(2)} \;=\; \sum_{\alpha=1}^{n-1} v_\alpha^{(1)} \otimes \left(v_\alpha^{(1)} - c_\alpha v_n^{(2)} \right)$$

which contradicts minimality of $n = \mathrm{rank}(t)$. The statement about $(v_\alpha^{(2)})$ is proven similarly. □

As a consequence we get a better upper bound

$$\mathrm{rank}(t) \leq \min \{\dim (V_1), \dim (V_2)\}.$$

Taking tensor products with the one-dimensional vector space $\mathbb{K}$ is in effect useless: for any vector space V we can canonically identify

$$\begin{aligned} V \otimes \mathbb{K} \;&\cong V \\ v \otimes \lambda \;&\mapsto \lambda v \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbb{K} \otimes V \;&\cong V \\ \lambda \otimes v \;&\mapsto \lambda v. \end{aligned}$$

By the obvious correspondence of bilinear maps $V_1 \times V_2 \to W$ and $V_2 \times V_1 \to W$, one also always gets a canonical identification

$$V_1 \otimes V_2 \;\cong\; V_2 \otimes V_1.$$

Almost equally obvious correspondences give the canonical identifications

$$(V_1 \otimes V_2) \otimes V_3 \;\cong\; V_1 \otimes (V_2 \otimes V_3)$$

etc., which allow us to omit parentheses in multiple tensor products.

A slightly more interesting property than the above obvious identifications is the embedding

$$V_2 \otimes V_1^* \hookrightarrow \mathrm{Hom}(V_1, V_2)$$

which is obtained by associating to $v_2 \otimes \varphi$ the linear map

$$v_1 \mapsto \varphi(v_1)\, v_2$$

(and extending linearly from the simple tensors to all tensors). In the exercises it is shown that when both V_1 and V_2 are finite dimensional, this gives a linear isomorphism $V_2 \otimes V_1^* \cong \mathrm{Hom}(V_1, V_2)$, and the rank of a tensor becomes the rank of a matrix of the corresponding linear map.

When

$$f : V_1 \to W_1 \qquad \text{and} \qquad g : V_2 \to W_2$$

are linear maps, then there is a linear map

$$f \otimes g : V_1 \otimes V_2 \to W_1 \otimes W_2$$

such that

$$(f \otimes g)(v_1 \otimes v_2) = f(v_1) \otimes g(v_2) \qquad \text{for all } v_1 \in V_1,\ v_2 \in V_2.$$

This clearly depends bilinearly on (f, g), so we get a canonical map

$$\mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2) \hookrightarrow \mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2),$$

which is easily seen to be injective. When all the vector spaces V_1, W_1, V_2, W_2 are finite dimensional, then the dimension of either side above is given by

$$\dim(V_1)\dim(V_2)\dim(W_1)\dim(W_2),$$

so in this case the canonical map is an isomorphism

$$\mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2) \cong \mathrm{Hom}(V_1 \otimes V_1, W_1 \otimes W_2).$$

As a particular case of the above, interpreting the dual of a vector space V as $V^* = \mathrm{Hom}(V, \mathbb{K})$ and using $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, we see that the tensor product of duals sits inside the dual of the tensor product. Explicitly, if V_1 and V_2 are vector spaces and $\varphi_1 \in V_1^*$, $\varphi_2 \in V_2^*$, then

$$v_1 \otimes v_2 \mapsto \varphi_1(v_1)\, \varphi_2(v_2)$$

defines an element of the dual of $V_1 \otimes V_2$. To summarize, we have an embedding

$$V_1^* \otimes V_2^* \hookrightarrow (V_1 \otimes V_2)^*.$$

If V_1 and V_2 are finite dimensional this gives the isomorphism

$$V_1^* \otimes V_2^* \cong (V_1 \otimes V_2)^*,$$

and later we will see that the fact that in infinite dimensional case we can only go to one direction, is essentially responsible for the asymmetry in the dualities between algebras and coalgebras.