4 On quantum groups

A building block of quantum groups

This section discusses a Hopf algebra H_q, which is an important building block of quantum groups — a kind of "quantum" version of a Borel subalgebra of the Lie algebra $\mathfrak{sl}_2$.

q-integers, q-factorials and q-binomial coefficients

For $n \in \mathbb{N}$ and $0 \leq k \leq n$, define the following rational (in fact polynomial) functions of q:

$$\text{the } q\text{-integer} \qquad [\![n]\!] = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q} \tag{4.1}$$

$$\text{the } q\text{-factorial} \qquad [\![n]\!]! = [\![1]\!]\,[\![2]\!] \cdots [\![n-1]\!]\,[\![n]\!] \tag{4.2}$$

$$\text{the } q\text{-binomial coefficient} \qquad \left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right] = \frac{[\![n]\!]!}{[\![k]\!]!\,[\![n-k]\!]!}, \tag{4.3}$$

and when $q \in \mathbb{C} \setminus \{0\}$, denote the values of these functions at q by

$$[\![n]\!]_q, \qquad [\![n]\!]_q!, \qquad \left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right]_q,$$

respectively.

Remark 1. *When $q = 1$, one recovers the usual integers, factorials and binomial coefficients.*

As simple special cases one has

$$[\![0]\!] = 0, \qquad [\![1]\!] = 1 \qquad \text{and} \qquad [\![0]\!]! = [\![1]\!]! = 1$$

and for all $n \in \mathbb{N}$

$$\left[\!\!\left[\begin{matrix} n \\ 0 \end{matrix} \right]\!\!\right] = \left[\!\!\left[\begin{matrix} n \\ n \end{matrix} \right]\!\!\right] = 1 \qquad \text{and} \qquad \left[\!\!\left[\begin{matrix} n \\ 1 \end{matrix} \right]\!\!\right] = \left[\!\!\left[\begin{matrix} n \\ n-1 \end{matrix} \right]\!\!\right] = [\![n]\!].$$

When q is a root of unity, degeneracies arise. Let p be the smallest positive integer such that $q^p = 1$. Then we have

$$[\![mp]\!]_q = 0 \quad \forall m \in \mathbb{N} \qquad \text{and} \qquad [\![n]\!]_q! = 0 \quad \forall n \geq p.$$

The values of the q-binomial coefficients at roots of unity are described as follows: if the quotients and remainders modulo p of n and k are $n = p\,D(n) + R(n)$ and $k = p\,D(k) + R(k)$ with $D(n), D(k) \in \mathbb{N}$ and $R(n), R(k) \in \{0, 1, 2, \ldots, p-1\}$, then

$$\left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right]_q = \binom{D(n)}{D(k)} \times \left[\!\!\left[\begin{matrix} R(n) \\ R(k) \end{matrix} \right]\!\!\right]_q.$$

In particular $\left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right]_q$ is non-zero only if the remainders modulo p of n and k satisfy $R(k) \leq R(n)$.

The Hopf algebra H_q

Let $q \in \mathbb{C} \setminus \{0\}$ and let H_q be the algebra with three generators a, a', b and relations

$$a\,a' = a'\,a = 1 \qquad , \qquad a\,b = q\,b\,a.$$

Because of the first relation we can write $a' = a^{-1}$ in H_q. The collection $(b^m\,a^n)_{m \in \mathbb{N},\, n \in \mathbb{Z}}$ is a vector space basis for H_q. The product in this basis is easily seen to be

$$\mu(b^{m_1} a^{n_1} \otimes b^{m_2} a^{n_2}) = q^{n_1 m_2}\, b^{m_1+m_2} a^{n_1+n_2}.$$

Exercise 1. *Show that there is a unique Hopf algebra structure on H_q such that the coproducts of a and b are given by*

$$\Delta(a) = a \otimes a \qquad \text{and} \qquad \Delta(b) = a \otimes b + b \otimes 1.$$

Show also that the following formulas hold in this Hopf algebra

$$\Delta(b^m a^n) = \sum_{k=0}^{m} \left[\!\left[\begin{matrix} m \\ k \end{matrix} \right]\!\right]_q b^k a^{m-k+n} \otimes b^{m-k} a^n \tag{4.4}$$

$$\epsilon(b^m a^n) = \delta_{m,0} \tag{4.5}$$

$$\gamma(b^m a^n) = (-1)^m \, q^{-m(m+1)/2-nm} \, b^m \, a^{-n-m}. \tag{4.6}$$

We will assume from here on that $q \neq 1$. Then the Hopf algebra H_q is clearly neither commutative nor cocommutative. In fact, H_q also serves as an example of a Hopf algebra where the antipode is not involutive: we have for example

$$\gamma(\gamma(b)) = -q^{-1} \gamma(ba^{-1}) = q^{-1} \, b \neq b.$$

About the restricted dual of H_q

Let us now consider the restricted dual H_q°. By Corollary 13 (Section 3), it is spanned by the representative forms of finite dimensional H_q-modules, so let us start concretely from low-dimensional modules.

Exercise 2. *Let A be an algebra and consider its restricted dual A°. Show that for a linear map $f : A \to \mathbb{C}$ the following are equivalent:*

- *The function f is a homomorphism of algebras.*
  *(*Remark: *This has the interpretation that f is a one-dimensional representation of A.)*
- *The element f is grouplike in the coalgebra A°.*

One-dimensional representations of H_q

Suppose $V = \mathbb{C}\, v$ is a one-dimensional H_q module with basis vector v. We have

$$a.v = z \, v$$

for some complex number z, which must be non-zero since $a \in H_q$ is invertible. Note that

$$a.(b.v) = q \, b.(a.v) = qz \, b.v,$$

which means that $b.v$ is an eigenvector of a with a different eigenvalue, $qz \neq z$. Eigenvectors corresponding to different eigenvalues would be linearly independent, so in the one dimensional module we must have $b.v = 0$. It is now straighforward to compute the action of $b^m a^n$,

$$b^m a^n.v = \delta_{m,0} \, z^n \, v,$$

from which we can read the only representative form $\lambda_{1,1} \in H_q^\circ$ in this case. We define $g_z \in H_q^\circ$ as the representative form

$$\langle g_z, b^m a^n \rangle = \delta_{m,0} \, z^n.$$

By the exercise above, the one-dimensional representations correspond to grouplike elements of H_q°, and indeed it is easy to verify by direct computation or as a special case of Equation (3.4) that

$$\mu^*(g_z) = g_z \otimes g_z.$$

To compute the products of two elements of this type, we use Equation (4.4):

$$\begin{aligned}\langle \Delta^*(g_z \otimes g_w), b^m a^n \rangle &= \langle g_z \otimes g_w, \Delta(b^m a^n) \rangle = \sum_{k=0}^{m} \left[\!\!\left[\begin{matrix} m \\ k \end{matrix} \right]\!\!\right]_q \langle g_z, b^k a^{m-k+n} \rangle \, \langle g_w, b^{m-k} a^n \rangle \\ &= \sum_{k=0}^{m} \left[\!\!\left[\begin{matrix} m \\ k \end{matrix} \right]\!\!\right]_q \delta_{k,0} \, \delta_{m-k,0} \, z^{m-k+n} \, w^n = \delta_{m,0} \, (zw)^n = \langle g_{zw}, b^m a^n \rangle,\end{aligned}$$

that is, the product in H_q° of these elements reads

$$\Delta^*(g_z \otimes g_w) = g_{zw}.$$

We see that the linear span of $(g_z)_{z \in \mathbb{C}\setminus\{0\}}$ in H_q° is isomorphic to the group algebra of the multiplicative group of non-zero complex numbers $\mathbb{C}[\mathbb{C}^*]$.

We also remark that there is the trivial one-dimensional representation, explicitly determined by Equation (4.5),

$$b^m a^n.v = \epsilon(b^m a^n) \, v = \delta_{m,0} \, v = \langle g_1, b^m a^n \rangle \, v,$$

and the corresponding grouplike element of the restricted dual is the unit of the restricted dual Hopf algebra, $g_1 = \epsilon^*(1) = 1_{H_q^\circ}$.

Two-dimensional representations of H_q

Let V be a two-dimensional H_q-module, and choose a basis v_1, v_2 in which a is in Jordan canonical form. Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ be the (different or equal) eigenvalues of a. Recall that if v is an eigenvector of a of eigenvalue z, then either $b.v = 0$ or $b.v$ is a nonzero eigenvector of a with eigenvalue qz. Let us now suppose that $q \neq -1$ (and as before we continue to assume $q \neq 0$ and $q \neq 1$) so that in the latter case b has to annihilate at least one eigenvectors of a and let us without loss of generality suppose that

$$a.v_1 = z_1 \, v_1 \qquad \text{and} \qquad b.v_1 = 0.$$

There are a few possible cases. Either a is diagonalizable or there is a size two Jordan block of a (in the latter case the eigenvalues of a must coincide), and either $b.v_2 = 0$ or $b.v_2$ is a nonzero multiple of v_1 (in which case we must have $z_1 = qz_2 \neq z_2$ by the above argument).

Consider first the case when a is diagonalizable and $b.v_2 = 0$. Then $a.v_2 = z_2 \, v_2$ and we easily compute

$$b^m a^n.v_1 = \delta_{m,0} \, z_1^n \, v_1 \qquad \text{and} \qquad b^m a^n.v_2 = \delta_{m,0} \, z_2^n \, v_2.$$

We read that the representative forms are of the same type as before,

$$\lambda_{1,1} = g_{z_1}, \qquad \lambda_{2,1} = 0, \qquad \lambda_{1,2} = 0, \qquad \lambda_{2,2} = g_{z_2}.$$

Consider then the case when a is not diagonalizable. We may suppose $a.v_1 = z_1 \, v_1$ and $a.v_2 = z_1 \, v_2 + v_1$. We observe like before that

$$(a - z_1)^2.v_2 = 0 \qquad \Rightarrow \qquad (a - qz_1)^2 b.v_2 = b(qa - qz_1)^2.v_2 = 0$$

which means that $b.v_2$ would have a generalized eigenvalue qz_1, which is impossible, so $b.v_2 = 0$, too. It is now easy to compute the action of the whole algebra on the module,

$$b^m a^n.v_1 = \delta_{m,0} \, z_1^n \, v_1 \qquad \text{and} \qquad b^m a^n.v_2 = \delta_{m,0} \left(z_1^n \, v_2 + n \, z_1^{n-1} \, v_1 \right).$$

Here we find one new representative form, we define $g'_z \in H_q^\circ$, for $z \in \mathbb{C} \setminus \{0\}$, by

$$\langle g'_z, b^m a^n \rangle = \delta_{m,0} \, n \, z^{n-1}.$$

Then the representative forms are

$$\lambda_{1,1} = g_{z_1}, \qquad \lambda_{2,1} = 0, \qquad \lambda_{1,2} = g'_{z_1}, \qquad \lambda_{2,2} = g_{z_1}.$$

The coproduct in H_q° of the newly found element can be read from Equation (3.4) with the result

$$\mu^*(g'_z) \; = \; g_z \otimes g'_z + g'_z \otimes g_z.$$

This could of course also be verified by the following direct calculation

$$\begin{aligned} \langle \mu^*(g'_z), b^{m_1} a^{n_1} \otimes b^{m_2} a^{n_2} \rangle &= \langle g'_z, b^{m_1} a^{n_1} b^{m_2} a^{n_2} \rangle \; = \; q^{n_1 m_2} \, \langle g'_z, b^{m_1+m_2} a^{n_1+n_2} \rangle \\ &= q^{n_1 m_2} \, \delta_{m_1+m_2,0} \, (n_1 + n_2) \, z^{n_1+n_2-1} \\ &= q^0 \, \delta_{m_1,0} \, \delta_{m_2,0} \left(n_1 \, z^{n_1-1} \, z^{n_2} + z^{n_1} \, n_2 \, z^{n_2-1} \right) \\ &= \langle g'_z \otimes g_z + g_z \otimes g'_z, b^{m_1} a^{n_1} \otimes b^{m_2} a^{n_2} \rangle. \end{aligned}$$

Consider finally the case when a is diagonalizable and $b.v_2$ is a nonzero multiple of v_1. As was shown earlier, this requires $z_1 = q z_2$, and we may assume a normalization of the basis vectors such that

$$a.v_1 \; = \; q z_2 \, v_1, \qquad a.v_2 \; = \; z_2 \, v_2, \qquad b.v_1 \; = \; 0, \qquad b.v_2 \; = \; v_1.$$

We then have

$$b^m a^n.v_1 \; = \; \delta_{m,0} \, (q z_2)^n \, v_1 \qquad \text{and} \qquad b^m a^n.v_2 \; = \; \delta_{m,0} \, z_2^n \, v_2 + \delta_{m,1} \, z_2^n \, v_1,$$

so we find one new representative form again. Defining $h_z^{(1)} \in H_q^\circ$, for $z \in \mathbb{C} \setminus \{0\}$, by

$$\langle h_z^{(1)}, b^m a^n \rangle \; = \; \delta_{m,1} \, z^n,$$

the representative forms in this case read

$$\lambda_{1,1} = g_{q z_2}, \qquad \lambda_{2,1} = 0, \qquad \lambda_{1,2} = h_{z_2}^{(1)}, \qquad \lambda_{2,2} = g_{z_2}.$$

From Equation (3.4) we get the coproduct

$$\mu^*(h_z^{(1)}) \; = \; g_{qz} \otimes h_z^{(1)} + h_z^{(1)} \otimes g_z.$$

Since H_q° is also a Hopf algebra, we would want to know also products of the newly found elements. We will only make explicit the subalgebra generated by $h_z^{(1)}$, leaving it as an exercise to compute the products for elements g'_z. It will turn out useful to define for any $k \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0\}$ the elements $h_z^{(k)}$ of the dual by

$$\langle h_z^{(k)}, b^m a^n \rangle \; = \; \delta_{m,k} \, z^n,$$

of which we have considered the special cases $h_z^{(1)}$ and $h_z^{(0)} = g_z$. The products are calculated as follows, using Equation (4.4),

$$\begin{aligned} \langle \Delta^*(h_z^{(k)} \otimes h_w^{(l)}), b^m a^n \rangle &= \langle h_z^{(k)} \otimes h_w^{(l)}, \Delta(b^m a^n) \rangle \; = \; \sum_{j=0}^{m} \left[\!\left[\begin{matrix} m \\ j \end{matrix} \right]\!\right]_q \langle h_z^{(k)}, b^j a^{m-j+n} \rangle \, \langle h_w^{(l)}, b^{m-j} a^n \rangle \\ &= \sum_{j=0}^{m} \left[\!\left[\begin{matrix} m \\ j \end{matrix} \right]\!\right]_q \delta_{j,k} \, z^{m-j+n} \, \delta_{m-j,l} \, w^n \; = \; \left[\!\left[\begin{matrix} k+l \\ k \end{matrix} \right]\!\right]_q z^{l+n} \, w^n \\ &= z^l \left[\!\left[\begin{matrix} k+l \\ k \end{matrix} \right]\!\right]_q \langle h_{zw}^{(k+l)}, b^m a^n \rangle, \end{aligned}$$

with the result

$$h_z^{(k)} \, h_w^{(l)} \; = \; z^l \left[\!\left[\begin{matrix} k+l \\ k \end{matrix} \right]\!\right]_q h_{zw}^{(k+l)}. \tag{4.7}$$

We are ready to prove the following.

Lemma 1. *Let $q \in \mathbb{C} \setminus \{0\}$ be such that $q^N \neq 1$ for all $N \in \mathbb{Z} \setminus \{0\}$. Then the algebra H_q can be embedded to its restricted dual by the linear map such that $b^m a^n \mapsto \tilde{b}^m \tilde{a}^n$, where*

$$\tilde{a} = g_q \qquad \text{and} \qquad \tilde{b} = h_1^{(1)}.$$

This embedding is an injective homomorphism of Hopf algebras.

Proof. First, $\tilde{a}$ is grouplike and thus invertible. Second, one sees from Equation (4.7) that

$$\tilde{a}\tilde{b} = q\, h_q^{(1)} = q\, \tilde{b}\tilde{a},$$

which shows that the needed relations are satisfied, and the given embedding is an algebra homomorphism. Denote by $\tilde{1} = \epsilon^*(1) = g_1$ the unit of the restricted dual Hopf algebra. From the earlier formulas we also read that

$$\mu^*(\tilde{b}) = \tilde{a} \otimes \tilde{b} + \tilde{b} \otimes \tilde{1},$$

and by the earlier exercise the values of the coproduct μ^*, the counit η^* and the antipode γ^* on the elements $\tilde{b}^m \tilde{a}^n$ are uniquely determined by these conditions. Finally, the images of the basis elements can be computed using Equation (4.7), and we get

$$\tilde{b}^m \tilde{a}^n = [\![m]\!]_q!\, h_{q^n}^{(m)},$$

which are non-zero and linearly independent elements of the dual when q is not a root of unity, so the embedding is indeed injective. □

Braided bialgebras and braided Hopf algebras

In this section we discuss the braiding structure that is characteristic of quantum groups in addition to just Hopf algebra structure.

Recall that if A is a bialgebra, then we're able to equip tensor products of A-modules with an A-module structure, and the one dimensional vector space $\mathbb{C}$ with a trivial module structure. The coalgebra axioms guarantee in addition that the canonical vector space isomorphisms

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$

and

$$V \otimes \mathbb{C} \cong V \cong \mathbb{C} \otimes V$$

become isomorphisms of A-modules. If A is cocommutative, $\Delta^{\mathrm{op}} = \Delta$, and if V and W are A-modules, then also the tensor flip

$$S_{V,W} : V \otimes W \to W \otimes V$$

becomes an isomorphism of A-modules. The property "braided" is a generalization of "cocommutative": we will not require equality of the coproduct and opposite coproduct, but only ask the two to be related by conjugation, and we will be able to keep weakened forms of some of the good properties of cocommutative bialgebras — in particular we obtain natural A-module isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V$$

that "braid" the tensor components.

The braid groups

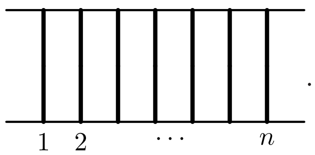
Let us start by what braiding usually means.

Definition 1. *For n a positive integer, the* <u>braid group</u> *on n strands is the group B_n with generators $\sigma_1, \sigma_2 \ldots, \sigma_{n-1}$ and relations*

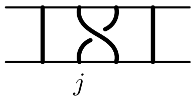
$$\sigma_j \, \sigma_{j+1} \, \sigma_j = \sigma_{j+1} \, \sigma_j \, \sigma_{j+1} \qquad \text{for } 1 \leq j < n \tag{4.8}$$

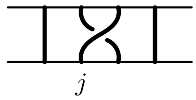
$$\sigma_j \, \sigma_k = \sigma_k \, \sigma_j \qquad \text{for } |j-k| > 1. \tag{4.9}$$

To see why this is called braiding, we visualize the elements as operations on n vertical strands, which we draw next to each other from bottom to top



The operations are continuation of the strands from the top, the generators and their inverses being visualized as follows

σ_j = 

and σ_j^{-1} = 

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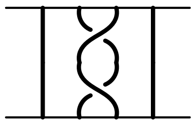
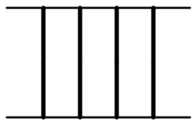
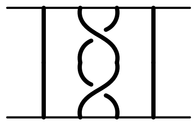
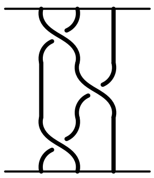
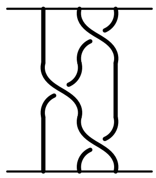
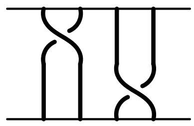
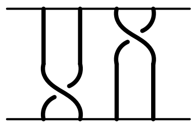
Having visualized the generators in this way, the equations $\sigma_j^{-1} \, \sigma_j = e = \sigma_j \, \sigma_j^{-1}$ tell us to identify the following kinds of pictures

 =  =  .

The braid group relations tell us to identify pictures for example as shown below

 $\overset{(4.8)}{=}$  and  $\overset{(4.9)}{=}$  .

Remark 2. *In the symmetric group S_n, the transpositions of consequtive elements satisfy the relations (4.8) and (4.9). Such transpositions generate S_n, so there exists a surjective group homomorphism $B_n \to S_n$ such that $\sigma_j \mapsto (j \;\; j+1)$. In other words, the symmetric group is isomorphic to a quotient of the braid group. In this quotient one only keeps track of the endpoints of the strands (permutation), forgetting about their possible entanglement (braid).*

The Yang-Baxter equation

A collection of complex numbers $(r^{k,l}_{i,j})_{i,j,k,l \in \{1,2,\ldots,d\}}$ is said to satisfy the Yang-Baxter equation if

$$\sum_{a,b,c=1}^{d} r^{l,m}_{a,b} \, r^{b,n}_{c,k} \, r^{a,c}_{i,j} = \sum_{a,b,c=1}^{d} r^{m,n}_{a,b} \, r^{l,a}_{i,c} \, r^{c,b}_{j,k} \qquad \text{for all } i,j,k,l,m,n \in \{1,2,\ldots,d\}. \tag{YBE}$$

Observe that there are d^6 equations imposed on d^4 complex numbers.

Let V be a vector space with basis $(v_i)_{i=1}^d$ and define

$$\check{R} : V \otimes V \to V \otimes V \qquad \text{by} \qquad \check{R}(v_i \otimes v_j) \; = \; \sum_{k,l=1}^{d} r_{i,j}^{k,l} \, v_k \otimes v_l.$$

Then the Yang-Baxter equation is equivalent with

$$\check{R}_{12} \circ \check{R}_{23} \circ \check{R}_{12} \; = \; \check{R}_{23} \circ \check{R}_{12} \circ \check{R}_{23}, \tag{4.10}$$

where $\check{R}_{12}, \check{R}_{23} : V \otimes V \otimes V \to V \otimes V \otimes V$ are defined as $\check{R}_{12} = \check{R} \otimes \mathrm{id}_V$ and $\check{R}_{23} = \mathrm{id}_V \otimes \check{R}$. This equation has obvious resemblance with Equation (4.8).

Example 1. *If we set $\check{R}(v_i \otimes v_j) = v_i \otimes v_j$ for all $i, j \in \{1, 2, \ldots, d\}$, that is $r_{i,j}^{k,l} = \delta_{i,k}\delta_{j,l}$ and $\check{R} = \mathrm{id}_{V\otimes V}$, then $\check{R}$ satisfies the Yang-Baxter equation since both sides of Equation (4.10) become $\mathrm{id}_{V\otimes V\otimes V}$.*

Example 2. *If we set $\check{R}(v_i \otimes v_j) = v_j \otimes v_i$ for all $i, j \in \{1, 2, \ldots, d\}$, that is $r_{i,j}^{k,l} = \delta_{i,l}\delta_{j,k}$ and $\check{R} = S_{V,V}$, then $\check{R}$ satisfies the Yang-Baxter equation, as is verified by the following calculation on simple tensors*

$$\begin{array}{ccccccc}
v_i \otimes v_j \otimes v_k & \overset{S_{V,V}\otimes \mathrm{id}_V}{\mapsto} & v_j \otimes v_i \otimes v_k & \overset{\mathrm{id}_V\otimes S_{V,V}}{\mapsto} & v_j \otimes v_k \otimes v_i & \overset{S_{V,V}\otimes \mathrm{id}_V}{\mapsto} & v_k \otimes v_j \otimes v_i \\
v_i \otimes v_j \otimes v_k & \overset{\mathrm{id}_V\otimes S_{V,V}}{\mapsto} & v_i \otimes v_k \otimes v_j & \overset{S_{V,V}\otimes \mathrm{id}_V}{\mapsto} & v_k \otimes v_i \otimes v_j & \overset{\mathrm{id}_V\otimes S_{V,V}}{\mapsto} & v_k \otimes v_j \otimes v_i.
\end{array}$$

Exercise 3. *Let $q \in \mathbb{C} \setminus \{0\}$ and set*

$$\check{R}(v_i \otimes v_j) = \begin{cases} q\, v_i \otimes v_j & \text{if } i = j \\ v_j \otimes v_i & \text{if } i < j \\ v_j \otimes v_i + (q - q^{-1})\, v_i \otimes v_j & \text{if } i > j \end{cases}.$$

Verify that $\check{R}$ satisfies the Yang-Baxter equation. Show also that $(\check{R} - q) \circ (\check{R} + q^{-1}) = 0$.

The notations $\check{R}_{12} = \check{R} \otimes \mathrm{id}_V$ and $\check{R}_{23} = \mathrm{id}_V \otimes \check{R}$ are a special case of acting on chosen tensor components. More generally, if $T : V \otimes V \to V \otimes V$ is a linear map, then on

$$V^{\otimes n} \; = \; \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$$

we define T_{ij}, $1 \leq i, j \leq n$, $i \neq j$, as the linear map that acts as T on the i^{th} and j^{th} tensor components and as identity on the rest. Explicitly, if

$$T(v_k \otimes v_{k'}) \; = \; \sum_{l,l'=1}^{d} t_{k,k'}^{l,l'} \, v_l \otimes v_{l'},$$

then (assuming $i < j$ for definiteness)

$$T_{ij}(v_{k_1} \otimes \cdots \otimes v_{k_n}) \; = \; \sum_{l,l'=1}^{d} t_{k_i,k_j}^{l,l'} \, v_{k_1} \otimes \cdots \otimes v_{k_{i-1}} \otimes v_l \otimes v_{k_{i+1}} \otimes \cdots \otimes v_{k_{j-1}} \otimes v_{l'} \otimes v_{k_{j+1}} \otimes \cdots \otimes v_{k_n}.$$

As an exercise, the reader can check that if we set $R = S_{V,V} \circ \check{R}$, then Equation (4.10) is equivalent with the equation

$$R_{12} \circ R_{13} \circ R_{23} \; = \; R_{23} \circ R_{13} \circ R_{12}. \tag{4.11}$$

Note that here we could have taken the following as definitions

$$\begin{aligned}
R_{12} &= R \otimes \mathrm{id}_V \\
R_{23} &= \mathrm{id}_V \otimes R \\
R_{13} &= (\mathrm{id}_V \otimes S_{V,V}) \circ (R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes S_{V,V}).
\end{aligned}$$

Proposition 2. *If $\check{R} : V \otimes V \to V \otimes V$ is bijective and satisfies the Yang-Baxter equation (4.10), then on $V^{\otimes n}$ there is a representation of the braid group B_n, $\rho : B_n \to \mathrm{Aut}(V^{\otimes n})$, such that $\rho(\sigma_j) = \check{R}_{j\,j+1}$ for all $j = 1, 2, \ldots, n-1$.*

Proof. First, if $\check{R}$ is bijective, then clearly $\check{R}_{j\,j+1} \in \mathrm{Aut}(V^{\otimes n})$ for all j. To show existence of a group homomorphism ρ with the given values on the generators σ_j, we need to verify the relations (4.8) and (4.9) for the images $\check{R}_{j\,j+1}$. The first relation follows from the Yang-Baxter equation (4.10), and the second is obvious since when $|j-k| > 1$, the matrix $\check{R}_{j\,j+1}$ acts as identity on the tensor components k and $k+1$, and vice versa. □

Universal R-matrix and braided bialgebras

The universal R-matrix is an element of algebra, which in representations becomes a solution to the Yang-Baxter equation. Let A be an algebra (in what follows always in fact a bialgebra or a Hopf algebra), and equip $A \otimes A$ and $A \otimes A \otimes A$ as usually with the algebra structures given by componentwise products, for example $(a \otimes a')\,(b \otimes b') = ab \otimes a'b'$ for all $a, a', b, b' \in A$. Suppose that we have an element $R \in A \otimes A$, which we write as a sum of elementary tensors

$$R = \sum_{i=1}^{r} s_i \otimes t_i,$$

with some $s_i, t_i \in A$, $i = 1, 2, \ldots, r$. Then we use the notations

$$R_{12} = \sum_i s_i \otimes t_i \otimes 1_A \qquad R_{13} = \sum_i s_i \otimes 1_A \otimes t_i \qquad R_{23} = \sum_i 1_A \otimes s_i \otimes t_i.$$

Definition 2. *Let A be a bialgebra (or a Hopf algebra). An element $R \in A \otimes A$ is called a* <u>*universal R-matrix*</u> *for A if*

(R0) *R is invertible in the algebra $A \otimes A$*

(R1) *for all $x \in A$ we have $\Delta^{\mathrm{op}}(x) = R\,\Delta(x)\,R^{-1}$*

(R2) $(\Delta \otimes \mathrm{id}_A)(R) = R_{13}\,R_{23}$

(R3) $(\mathrm{id}_A \otimes \Delta)(R) = R_{13}\,R_{12}$.

A pair (A, R) as above is called a <u>*braided bialgebra*</u> *(or* <u>*braided Hopf algebra*</u>*, if A is a Hopf algebra).*

Instead of the terms "braided bialgebra" or "braided Hopf algebra", Drinfeld originally used the terms "quasitriangular bialgebra" and "quasitriangular Hopf algebra", which are therefore occasionally used in literature.

Example 3. *If A is a commutative bialgebra, $\Delta = \Delta^{\mathrm{op}}$, then $R = 1_A \otimes 1_A$ is a universal R-matrix. Thus braided bialgebras generalize cocommutative bialgebras.*

Exercise 4. *Let $A = \mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$ be the group algebra of the cyclic group of order N, generated by one element θ such that $\theta^N = 1_A$. Denote also $\omega = \exp(2\pi\mathrm{i}/N)$. Then the element*

$$R = \frac{1}{N} \sum_{k,l=0}^{N-1} \omega^{kl}\, \theta^k \otimes \theta^l$$

is a universal R-matrix for A. (Hint: To find the inverse element, Proposition 7 may help.)

Lemma 3. *If R is a universal R-matrix for a bialgebra A, then $R_{21}^{-1} = S_{A,A}(R^{-1})$ is also a universal R-matrix for A.*

Proof. Exercise. $\square$

The promised isomorphism of tensor product representations in two different orders goes as follows. Let A be a braided bialgebra (or braided Hopf algebra) with universal R-matrix $R \in A \otimes A$, and suppose that V and W are two A-modules, with $\rho_V : A \to \mathrm{End}(V)$ and $\rho_W : A \to \mathrm{End}(W)$ the respective representations. Recall that the vector spaces $V \otimes W$ and $W \otimes V$ are equipped with the representations of A given by $\rho_{V \otimes W} = (\rho_V \otimes \rho_W) \circ \Delta$ and $\rho_{W \otimes V} = (\rho_W \otimes \rho_V) \circ \Delta$.

Proposition 4. *The linear map $c_{V,W} : V \otimes W \to W \otimes V$ defined by*

$$c_{V,W} \;=\; S_{V,W} \circ \Big((\rho_V \otimes \rho_W)(R) \Big)$$

is an isomorphism of A-modules.

Proof. The map $(\rho_V \otimes \rho_W)(R) : V \otimes W \to V \otimes W$ is bijective, with inverse $(\rho_V \otimes \rho_W)(R^{-1})$. Since $S_{V,W} : V \otimes W \to W \otimes V$ is obviously also bijective, the map $c_{V,W}$ is indeed a bijective linear map. We must only show that it respects the A-module structures. Let $x \in A$ and compute

$$\begin{aligned}
c_{V,W} \circ \rho_{V \otimes W}(x) &= S_{V,W} \circ \Big((\rho_V \otimes \rho_W)(R) \Big) \circ \Big((\rho_V \otimes \rho_W)(\Delta(x)) \Big) \\
&= S_{V,W} \circ \Big((\rho_V \otimes \rho_W)(R \Delta(x)) \Big) \\
&\overset{(R1)}{=} S_{V,W} \circ \Big((\rho_V \otimes \rho_W)(\Delta^{\mathrm{op}}(x) R) \Big) \\
&= S_{V,W} \circ \Big((\rho_V \otimes \rho_W)(\Delta^{\mathrm{op}}(x)) \Big) \circ \Big((\rho_V \otimes \rho_W)(R) \Big) \\
&= \Big((\rho_W \otimes \rho_V)(\Delta(x)) \Big) \circ S_{V,W} \circ \Big((\rho_V \otimes \rho_W)(R) \Big) \\
&= \rho_{W \otimes V}(x) \circ c_{V,W}.
\end{aligned}$$

This shows that the action by x in the different modules is preserved by $c_{V,W}$. $\square$

A few more properties of braided bialgebras are listed in the following.

Proposition 5. *Let (A, R) be a braided bialgebra. Then we have*

- $(\epsilon \otimes \mathrm{id}_A)(R) = 1_A$ *and* $(\mathrm{id}_A \otimes \epsilon)(R) = 1_A$
- $R_{12} \, R_{13} \, R_{23} \;=\; R_{23} \, R_{13} \, R_{12}$.

Proof. To prove the second property, write

$$\begin{aligned}
R_{12} \, R_{13} \, R_{23} &\overset{(R2)}{=} R_{12} \, (\Delta \otimes \mathrm{id}_A)(R) \overset{(R1)}{=} (\Delta^{\mathrm{op}} \otimes \mathrm{id}_A)(R) \, R_{12} \\
&= \Big((S_{A,A} \otimes \mathrm{id}_A) \circ (\Delta \otimes \mathrm{id}_A)(R) \Big) R_{12} \overset{(R2)}{=} \Big((S_{A,A} \otimes \mathrm{id}_A)(R_{13} R_{23}) \Big) R_{12} \\
&= R_{23} \, R_{13} \, R_{12}.
\end{aligned}$$

To prove the first formula of the first property, we use two different ways to rewrite the expression

$$(\epsilon \otimes \mathrm{id}_A \otimes \mathrm{id}_A) \circ (\Delta \otimes \mathrm{id}_A)(R). \tag{4.12}$$

On the one hand, we could simply use $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$ to write (4.12) as R. On the other hand, if we denote $r = (\epsilon \otimes \mathrm{id}_B)(R) \in A$ and use property (R2) of R-matrices, we get

$$(4.12) \overset{(R2)}{=} (\epsilon \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(R_{13} R_{23}) \;=\; (1_A \otimes r) \, R.$$

The equality of the two simplified expressions, $R = (1_A \otimes r) R$, implies $1_A \otimes r = 1_A \otimes 1_A$ since R is invertible, and therefore we get $r = 1_A$ as claimed.

The case of $(\mathrm{id}_A \otimes \epsilon)(R)$ is handled similarly by considering the expression $(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon) \circ (\mathrm{id} \otimes \Delta)(R)$ instead of (4.12). $\square$

Suppose that $\rho_V : A \to \mathrm{End}(V)$ is a representation of a braided bialgebra A. The vector space $V^{\otimes n}$ is equipped with the representation of A

$$\rho \;=\; (\rho_V \otimes \rho_V \otimes \cdots \otimes \rho_V) \circ \Delta^{(n)},$$

where $\Delta^{(n)}$ denotes the $n-1$-fold coproduct, defined (for example) as

$$\Delta^{(n)} \;=\; (\Delta \otimes \mathrm{id}_A \otimes \cdots \otimes \mathrm{id}_A) \circ \cdots \circ (\Delta \otimes \mathrm{id}_A) \circ \Delta,$$

although by coassociativity we are allowed to write it in other ways if we wish. Combining the second property of Proposition 5, the observation that Equations (4.11) and (4.10) are equivalent, and Proposition 4, we have proved the following.

Theorem 6. *Let A be a braided bialgebra (or a braided Hopf algebra) with universal R-matrix $R \in A \otimes A$, and let $\rho_V : A \to \mathrm{End}(V)$ be a representation of A in a vector space V. Then the linear map $\check{R} : V \otimes V \to V \otimes V$ given by*

$$\check{R} \;=\; c_{V,V} \;=\; S_{V,V} \circ \Big((\rho_V \otimes \rho_V)(R) \Big)$$

is a solution to the Yang-Baxter equation. Moreover, on the n-fold tensor product space $V^{\otimes n}$, the braid group action defined by $\check{R}$ as in Proposition 2 commutes with the action of A.

Braided Hopf algebras

Let $A = (A, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra, and suppose that there exists a universal R-matrix $R \in A \otimes A$, i.e. that A can be made a braided Hopf algebra. We will now investigate some implications that this has on the structure of the Hopf algebra and on the universal R-matrix..

Proposition 7. *For a braided Hopf algebra A we have*

$$(\gamma \otimes \mathrm{id}_A)(R) \;=\; R^{-1} \qquad \text{and} \qquad (\gamma \otimes \gamma)(R) \;=\; R.$$

Proof. Exercise. (Hint: For the first statement, remember Proposition 5. For the second, Lemma 3 may come in handy.) □

We can now prove a statement analogous to the property that in cocommutative Hopf algebras the square of the antipode is the identity. Here we obtain that the square of the antipode of a braided Hopf algebra is an inner automorphism.

Theorem 8. *Let A be a braided Hopf algebra with a universal R-matrix $R = \sum_i s_i \otimes t_i$. Then, for all $x \in A$ we have*

$$\gamma(\gamma(x)) \;=\; u\, x\, u^{-1},$$

where $u \in A$ is

$$u \;=\; \mu \circ (\gamma \otimes \mathrm{id}_A) \circ S_{A,A}(R) \;=\; \sum_i \gamma(t_i) s_i.$$

We also have

$$\gamma^{-1}(x) \;=\; u^{-1}\, \gamma(x)\, u.$$

Proof. We will first prove an auxiliary formula, $\sum_{(x)} \gamma(x_{(2)})\, u\, x_{(1)} = \epsilon(x)\, u$ for all $x \in A$. To get this, calculate

$$\begin{aligned}
\sum_{(x)} \gamma(x_{(2)})\, u\, x_{(1)} &= \sum_{(x)} \sum_i \gamma(x_{(2)})\gamma(t_i) s_i x_{(1)} = \sum_{(x)} \sum_i \gamma(t_i x_{(2)}) s_i x_{(1)} \\
&= \sum_{(x)} \sum_i \mu \circ (\gamma \otimes \mathrm{id}_A)\Big(x_{(2)} t_i \otimes s_i x_{(1)}\Big) \\
&= \mu \circ (\gamma \otimes \mathrm{id}_A) \circ S_{A,A}\Big(R\Delta(x)\Big) \\
&\overset{(R1)}{=} \mu \circ (\gamma \otimes \mathrm{id}_A) \circ S_{A,A}\Big(\Delta^{\mathrm{op}}(x) R\Big) \\
&= \sum_{(x)} \sum_i \gamma(x_{(1)} t_i) x_{(2)} s_i = \sum_{(x)} \sum_i \gamma(t_i)\gamma(x_{(1)}) x_{(2)} s_i \\
&\overset{(H3)}{=} \epsilon(x) \sum_i \gamma(t_i)\, 1_A\, s_i = \epsilon(x)\, u.
\end{aligned}$$

We will then show that $\gamma(\gamma(x))u = ux$. To do this, use the auxiliary formula for the first component "$x_{(1)}$" of the coproduct of $x \in A$ in the third equality below

$$\begin{aligned}
\gamma(\gamma(x))\, u &\overset{(H2')}{=} \gamma(\gamma(\sum_{(x)} \epsilon(x_{(1)})\, x_{(2)}))\, u = \sum_{(x)} \gamma(\gamma(x_{(2)}))\, \epsilon(x_{(1)})\, u \\
&\overset{\text{auxiliary}}{=} \sum_{\substack{(x) \\ \text{double coproduct}}} \gamma(\gamma(x_{(3)}))\, \gamma(x_{(2)})\, u\, x_{(1)} \\
&= \sum_{\substack{(x) \\ \text{double coproduct}}} \gamma(x_{(2)} \gamma(x_{(3)}))\, u\, x_{(1)} \\
&\overset{(H3)}{=} \sum_{(x)} \epsilon(x_{(2)})\, \gamma(1_A)\, u\, x_{(1)} \\
&\overset{(H2')}{=} \gamma(1_A)\, u\, x = u\, x.
\end{aligned}$$

Now to prove the formula $\gamma(\gamma(x)) = uxu^{-1}$ it suffices to show that u is invertible. We claim that the inverse is

$$\tilde{u} = \mu \circ (\mathrm{id}_A \otimes \gamma^2) \circ S_{A,A}(R) = \sum_i t_i \gamma^2(s_i).$$

Let us calculate, using the property $\gamma^2(x)u = ux$,

$$\begin{aligned}
\tilde{u}\, u &= \sum_i t_i\, \gamma^2(s_i)\, u = \sum_i t_i\, u\, s_i = \sum_{i,j} t_i\, \gamma(t_j)\, s_j\, s_i \\
&\overset{\text{Prop 7}}{=} \sum_{i,j} \gamma(t_i)\, \gamma(t_j)\, s_j\, \gamma(s_i) = \sum_{i,j} \gamma(t_j t_i)\, s_j\, \gamma(s_i) \\
&= \mu^{\mathrm{op}} \circ (\mathrm{id}_A \otimes \gamma)\Big(\sum_{i,j} s_j \gamma(s_i) \otimes t_j t_i\Big) \\
&\overset{\text{Prop 7}}{=} \mu^{\mathrm{op}} \circ (\mathrm{id}_A \otimes \gamma)\Big(R\, R^{-1}\Big) = \gamma(1_A) 1_A = 1_A.
\end{aligned}$$

Likewise,

$$u\, \tilde{u} = \sum_i u\, t_i\, \gamma^2(s_i) = \sum_i \gamma^2(t_i)\, u\, \gamma^2(s_i),$$

which by Proposition 7 equals $\sum_i t_i\, u\, s_i$, and this expression was already computed above to be 1_A.

It is an exercise to derive the last statement from the first. □

The Drinfeld double construction

There is a systematic way of creating braided Hopf algebras, the Drinfeld double construction.

Let us assume that $A = (A, \mu, \Delta, \eta, \epsilon, \gamma)$ is a Hopf algebra such that γ has an inverse γ^{-1}, and $B \subset A^\circ$ is a sub-Hopf algebra. We denote the unit of A simply by $1 = \eta(1)$, and the unit of A° (and thus also of B) by 1^*. Thus for any $a \in A$ we have $\langle 1^*, a\rangle = \epsilon(a)$. For any $\varphi \in B$ we use the following notation for the coproduct

$$\mu^*(\varphi) \;=\; \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)}.$$

Theorem 9. *Let A and $B \subset A^\circ$ be as above. Then the space $A \otimes B$ admits a unique Hopf algebra structure such that:*

(i) The map $\iota_A : A \to A \otimes B$ given by $a \mapsto a \otimes 1^*$ is a homomorphism of Hopf algebras.

(ii) The map $\iota_B : B^{\mathrm{cop}} \to A \otimes B$ given by $\varphi \mapsto 1 \otimes \varphi$ is a homomorphism of Hopf algebras.

(iii) For all $a \in A$, $\varphi \in B$ we have

$$(a \otimes 1^*)\,(1 \otimes \varphi) \;=\; a \otimes \varphi.$$

(iv) For all $a \in A$, $\varphi \in B$ we have

$$(1 \otimes \varphi)\,(a \otimes 1^*) \;=\; \sum_{(a)} \sum_{(\varphi)} \langle \varphi_{(1)}, a_{(3)}\rangle\, \langle \varphi_{(3)}, \gamma^{-1}(a_{(1)})\rangle\, a_{(2)} \otimes \varphi_{(2)}.$$

This Hopf algebra is denoted by $\mathcal{D}(A, B)$ and it is called the Drinfeld double associated to A and B.

Example 4. *When A is finite dimensional, $A^\circ = A^*$ is a Hopf algebra. It can also be shown that the antipode is always invertible in the finite dimensional case. The Drinfeld double associated to A and A^* is then denoted simply by $\mathcal{D}(A)$.*

Example 5. *When q is not a root of unity, we showed in Lemma 1 that the Hopf algebra H_q can be embedded to its restricted dual by a map such that $a \mapsto \tilde{a}$, $b \mapsto \tilde{b}$. We will later consider in detail the Drinfeld double associated to the Hopf algebra H_q and the sub-Hopf algebra of H_q° which is isomorphic to H_q.*

Proof of uniqueness in Theorem 9. When one claims that something exists and is uniquely determined by some given properties, it is often convenient to start with a proof of uniqueness, in the course of which one obtains explicit formulas that help proving existence. This is what we will do now. Denote the structural maps of $\mathcal{D}(A, B)$ by $\mu_{\mathcal{D}}$, $\Delta_{\mathcal{D}}$, $\eta_{\mathcal{D}}$, $\epsilon_{\mathcal{D}}$ and $\gamma_{\mathcal{D}}$, in order to avoid confusion with the structural maps of A (and of A°).

In order to prove that the product $\mu_{\mathcal{D}}$ is uniquely determined by the conditions, it suffices to compute its values on simple tensors. So let $a, b \in A$, $\varphi, \psi \in B$ and use the propertiy (iii) to write $(a \otimes \varphi) = (a \otimes 1^*)(1 \otimes \varphi)$ and $(b \otimes \psi) = (b \otimes 1^*)(1 \otimes \psi)$. Then calculate, assuming that $\mu_{\mathcal{D}}$ is an associative product,

$$\begin{aligned}
(a \otimes \varphi)\,(b \otimes \psi) \;&=\; (a \otimes 1^*)\,(1 \otimes \varphi)\,(b \otimes 1^*)\,(1 \otimes \psi) \\
&\overset{\text{(iv)}}{=} (a \otimes 1^*)\Big(\sum_{(\varphi),(b)} \langle \varphi_{(1)}, b_{(3)}\rangle\,\langle \varphi_{(3)}\ \gamma^{-1}(b_{(1)})\rangle\, b_{(2)} \otimes \varphi_{(2)}\Big)\,(1 \otimes \psi) \\
&\overset{\text{(iii)}}{=} (a \otimes 1^*)\Big(\sum_{(\varphi),(b)} \langle \varphi_{(1)}, b_{(3)}\rangle\,\langle \varphi_{(3)}, \gamma^{-1}(b_{(1)})\rangle\,(b_{(2)} \otimes 1^*)\,(1 \otimes \varphi_{(2)})\Big)\,(1 \otimes \psi) \\
&\overset{\text{(i) and (ii)}}{=} \sum_{(\varphi),(b)} \langle \varphi_{(1)}, b_{(3)}\rangle\,\langle \varphi_{(3)}, \gamma^{-1}(b_{(1)})\rangle\,(ab_{(2)} \otimes 1^*)\,(1 \otimes \varphi_{(2)}\psi).
\end{aligned}$$

By (iii) this simplifies to

$$(a \otimes \varphi)\,(b \otimes \psi) \;=\; \sum_{(\varphi),(b)} \langle \varphi_{(1)}, b_{(3)}\rangle\,\langle \varphi_{(3)}, \gamma^{-1}(b_{(1)})\rangle\, ab_{(2)} \otimes \varphi_{(2)}\psi, \tag{4.13}$$

and we see that the product $\mu_{\mathcal{D}}$ is indeed uniquely determined.

The unit in an associative algebra is always uniquely determined, and it is easy to show that it has to be given by

$$\eta_{\mathcal{D}}(1) \;=\; 1 \otimes 1^*. \tag{4.14}$$

The coproduct has to be a homomorphism of algebras. Thus using (iii): $(a \otimes \varphi) = (a \otimes 1^*)(1 \otimes \varphi) = \iota_A(a)\,\iota_B(\varphi)$, and (i): $\Delta_{\mathcal{D}}(\iota_A(a)) = \sum_{(a)} \iota_A(a_{(1)}) \otimes \iota_A(a_{(2)})$ and (ii): $\Delta_{\mathcal{D}}(\iota_B(\varphi)) = \sum_{(\varphi)} \iota_B(\varphi_{(2)}) \otimes \iota_B(\varphi_{(1)})$ we get

$$\Delta_{\mathcal{D}}(a \otimes \varphi) \;=\; \sum_{(a),(\varphi)} (a_{(1)} \otimes \varphi_{(2)}) \otimes (a_{(2)} \otimes \varphi_{(1)}). \tag{4.15}$$

The counit, too, has to be a homomorphism of algebras, so as above we easily get

$$\epsilon_{\mathcal{D}}(a \otimes \varphi) \;=\; \epsilon(a)\,\langle \varphi, 1\rangle. \tag{4.16}$$

Finally, the antipode has to be a homomorphism of algebras from $\mathcal{D}(A,B)$ to $\mathcal{D}(A,B)^{\mathrm{op}}$, so again by (iii) we must have $\gamma_{\mathcal{D}}(a \otimes \varphi) = \gamma(\iota_B(\varphi))\gamma(\iota_A(a))$. From (i) we get the obvious $\gamma(\iota_A(a)) = \gamma(a) \otimes 1^*$. Recall that the antipode of the co-opposite Hopf algebra is the inverse of the ordinary, and that the antipode of the restricted dual is obtained by taking the transpose. Then (ii) gives $\gamma_{\mathcal{D}}(\iota_B(\varphi)) = 1 \otimes (\gamma^*)^{-1}(\varphi)$. Now using (iv) and the homomorphism properties of antipodes, and properties of transpose, we get

$$\gamma_{\mathcal{D}}(a \otimes \varphi) \;=\; \sum_{(a),(\varphi)} \langle \varphi_{(1)}, \gamma^{-1}(a_{(3)})\rangle\,\langle \varphi_{(3)}, a_{(1)}\rangle\;\gamma(a_{(2)}) \otimes (\gamma^*)^{-1}(\varphi_{(2)}). \tag{4.17}$$

□

Before we turn to verifying the existence part of the proof, let us already show how the Drinfeld double construction yields braided Hopf algebras. Assume here that A is a finite dimensional Hopf algebra with basis $(e_i)_{i=1}^d$, and let $(\delta^i)_{i=1}^d$ denote the dual basis for A^*, so that

$$\langle \delta^i, e_j\rangle \;=\; \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We have already met the evaluation map $A^* \otimes A \to \mathbb{C}$ given by $\varphi \otimes a \mapsto \langle \varphi, a\rangle$. Let us now introduce the coevaluation map $\mathrm{coev} : \mathbb{C} \to A \otimes A^*$, which under the identification $A \otimes A^* \cong \mathrm{Hom}(A,A)$ corresponds to $\lambda \mapsto \lambda\, \mathrm{id}_A$. We can write explicitly

$$\mathrm{coev}(\lambda) \;=\; \lambda \sum_{i=1}^d e_i \otimes \delta^i.$$

Below we will frequently use the formula

$$\sum_{i=1}^d \langle \delta^i, b\rangle\, e_i \;=\; b,$$

valid for any $b \in A$. The combination of counitality with the defining property of the antipode is repeatedly used abusing the notation for multiple coproducts, for example as

$$\sum_{(b)} \Big(\gamma(b_{(j)}) b_{(j+1)}\Big) \otimes b_{(1)} \otimes \cdots \otimes b_{(j-1)} \otimes b_{(j+2)} \otimes \cdots \otimes b_{(n)} \;=\; \sum_{(b)} \Big(1_A\Big) \otimes b_{(1)} \otimes \cdots \otimes b_{(n-2)},$$

and analoguously in other similar cases, also with γ^{-1} in the opposite or co-opposite cases.

Theorem 10. *Let A be a finite dimensional Hopf algebra with invertible antipode, and let $(e_i)_{i=1}^d$ and $(\delta^i)_{i=1}^d$ be dual bases of A and A^*. Then the Drinfeld double $\mathcal{D}(A)$ is a braided Hopf algebra with a universal R-matrix*

$$R \;=\; (\iota_A \otimes \iota_{A^*})(\mathrm{coev}(1)) \;=\; \sum_{i=1}^{d} (e_i \otimes 1^*) \otimes (1 \otimes \delta^i).$$

Proof. Let us start by showing (R0), i.e. that R is invertible. The inverse is given by

$$\bar{R} \;=\; \sum_i (\gamma(e_i) \otimes 1^*) \otimes (1 \otimes \delta^i),$$

as Proposition 7 requires. We compute

$$\begin{aligned} R\,\bar{R} \;&= \Big(\sum_i (e_i \otimes 1^*) \otimes (1 \otimes \delta^i)\Big) \Big(\sum_j (\gamma(e_j) \otimes 1^*) \otimes (1 \otimes \delta^j)\Big) \\ &= \sum_{i,j} (e_i \gamma(e_j) \otimes 1^*) \otimes (1 \otimes \delta^i \delta^j). \end{aligned}$$

We would like to show that this elements of $\mathcal{D}(A) \otimes \mathcal{D}(A) = A \otimes A^* \otimes A \otimes A^*$ is the unit $1_{\mathcal{D}} \otimes 1_{\mathcal{D}} = 1 \otimes 1^* \otimes 1 \otimes 1^*$. Consider evaluating the expressions in $A \otimes A^* \otimes A \otimes A^*$ in the second and fourth components at $b \in A$ and $c \in A$. By the above calculation we see that $R\,\bar{R}$ evaluates to

$$\begin{aligned} \sum_{i,j} e_i\gamma(e_j) \otimes 1 \,\langle 1^*, b\rangle\, \langle \delta^i \delta^j, c\rangle \;&= \epsilon(b) \sum_{i,j} \sum_{(c)} e_i\gamma(e_j) \otimes 1 \,\langle \delta^i, c_{(1)}\rangle\, \langle \delta^j, c_{(2)}\rangle \\ &= \epsilon(b) \sum_{(c)} c_{(1)}\gamma(c_{(2)}) \otimes 1 \;=\; \epsilon(b)\epsilon(c)\; 1 \otimes 1. \end{aligned}$$

But the unit $1 \otimes 1^* \otimes 1 \otimes 1^*$ would obviously have evaluated to the same value, so we conclude that $\bar{R}$ is a right inverse: $R\,\bar{R} = 1 \otimes 1^* \otimes 1 \otimes 1^*$. To show that $\bar{R}$ is a left inverse is similar.

To prove property (R1), note that the set of elements that verifies the property

$$S \;=\; \Big\{ x \in \mathcal{D}(A) \;:\; \Delta^{\mathrm{op}}_{\mathcal{D}}(x)\, R \;=\; R\, \Delta_{\mathcal{D}}(x) \Big\}$$

is a subalgebra of $\mathcal{D}$: indeed the unit $1_{\mathcal{D}} = 1 \otimes 1^*$ has coproduct $\Delta_{\mathcal{D}}(1_{\mathcal{D}}) = 1_{\mathcal{D}} \otimes 1_{\mathcal{D}} = \Delta^{\mathrm{op}}(1_{\mathcal{D}})$ so it is clear that $1_{\mathcal{D}} \in S$, and if $x, y \in S$, then

$$\Delta^{\mathrm{op}}_{\mathcal{D}}(xy)\, R \;=\; \Delta^{\mathrm{op}}_{\mathcal{D}}(x)\, \Delta^{\mathrm{op}}_{\mathcal{D}}(y)\, R \;=\; \Delta^{\mathrm{op}}_{\mathcal{D}}(x)\, R\, \Delta_{\mathcal{D}}(y) \;=\; R\, \Delta_{\mathcal{D}}(x)\, \Delta_{\mathcal{D}}(y) \;=\; R\, \Delta_{\mathcal{D}}(xy).$$

By the defining formulas for the products in a Drinfeld double, elements of the form $a \otimes 1^*$ and $1 \otimes \varphi$ generate $\mathcal{D}(A)$ as an algebra, so it suffices to show that the property (R1) holds for elements of these two forms.

Consider an element of the form $a \otimes 1^*$. We only need the easy product formulas in the Drinfeld double to compute

$$\begin{aligned} \Delta^{\mathrm{op}}_{\mathcal{D}}(a \otimes 1)\, R \;&= \sum_i \sum_{(a)} \Big((a_{(2)} \otimes 1^*)\,(e_i \otimes 1^*)\Big) \otimes \Big((a_{(1)} \otimes 1^*)\,(1 \otimes \delta^i)\Big) \\ &= \sum_i \sum_{(a)} \Big((a_{(2)} e_i \otimes 1^*)\Big) \otimes \Big((a_{(1)} \otimes \delta^i)\Big), \end{aligned}$$

but for the other term we need the more general products

$$\begin{aligned} R\, \Delta_{\mathcal{D}}(a \otimes 1) \;&= \sum_i \sum_{(a)} \Big((e_i \otimes 1^*)\,(a_{(1)} \otimes 1^*)\Big) \otimes \Big((1 \otimes \delta^i)\,(a_{(2)} \otimes 1^*)\Big) \\ &= \sum_i \sum_{(a)} \Big(\, e_i a_{(1)} \otimes 1^* \Big) \otimes \Big(\sum_{(\delta^i)} \Big\langle (\delta^i)_{(3)}, \gamma^{-1}(a_{(2)}) \Big\rangle \Big\langle (\delta^i)_{(1)}, a_{(4)} \Big\rangle\, a_{(3)} \otimes (\delta^i)_{(2)} \Big) \end{aligned}$$

To show the equality of these two expressions in $\mathcal{D} \otimes \mathcal{D} \cong A \otimes A^* \otimes A \otimes A^*$, evaluate in the second and fourth components on two elements b, c of A. The first expression evaluates to

$$\sum_i \sum_{(a)} \epsilon(b) \left\langle \delta^i, c \right\rangle a_{(2)} e_i \otimes a_{(1)} \;=\; \sum_{(a)} \epsilon(b)\; a_{(2)} c \otimes a_{(1)}$$

and the second to

$$\begin{aligned}
&\sum_i \sum_{(a)} \sum_{(\delta^i)} \epsilon(b) \left\langle (\delta^i)_{(2)}, c \right\rangle \left\langle (\delta^i)_{(3)}, \gamma^{-1}(a_{(2)}) \right\rangle \left\langle (\delta^i)_{(1)}, a_{(4)} \right\rangle e_i a_{(1)} \otimes a_{(3)} \\
=\; & \epsilon(b) \sum_i \sum_{(a)} \left\langle \delta^i, a_{(4)}\, c\, \gamma^{-1}(a_{(2)}) \right\rangle \; e_i a_{(1)} \otimes a_{(3)} \\
=\; & \epsilon(b) \sum_{(a)} a_{(4)}\, c\, \gamma^{-1}(a_{(2)})\, a_{(1)} \otimes a_{(3)} \overset{\text{(H3) for } A^{\mathrm{op}}}{=} \epsilon(b) \sum_{(a)} a_{(2)}\, c \otimes a_{(1)},
\end{aligned}$$

which is the same as the first. We conclude that for all $a \in A$ the equality $\Delta^{\mathrm{op}}(a \otimes 1^*) R = R \Delta(a \otimes 1^*)$ holds.

Showing that elements of the form $1 \otimes \varphi$ satisfy the property is similar. We have then shown that the set of elements S for which $\Delta^{\mathrm{op}}(x) R = R\Delta(x)$ holds is a subalgebra containing a generating set of elements, so $S = \mathcal{D}(A)$. Since we have also shown that R is invertible, we now conclude property (R1).

Properties (R2) and (R3) of the R-matrix are similar and they can be verified in the same way. We leave this as an exercise. □

We should still verify that in the Drinfeld double construction the structural maps $\mu_{\mathcal{D}}$, $\Delta_{\mathcal{D}}$, $\eta_{\mathcal{D}}$, $\epsilon_{\mathcal{D}}$ and $\gamma_{\mathcal{D}}$, given by formulas (4.13 – 4.17), satisfy the axioms (H1 – H6). This is mostly routine and we will leave checking some of the axioms to the dedicated reader. In view of formulas (4.13 – 4.17) it is also clear that the embedding maps $\iota_A : A \to \mathcal{D}(A, B)$ and $\iota_B : B^{\mathrm{cop}} \to \mathcal{D}(A, B)$ are homomorphisms of Hopf algebras.

For checking the associativity property we still introduce a lemma. Note that the product (4.13) can be written as

$$\mu_{\mathcal{D}} \;=\; (\mu \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B) \;:\; A \otimes B \otimes A \otimes B \to A \otimes B, \tag{4.18}$$

where $\tau : B \otimes A \to A \otimes B$ is given by

$$\tau(\varphi \otimes a) \;=\; \sum_{(a),(\varphi)} \langle \varphi_{(1)}, a_{(3)} \rangle \, \langle \varphi_{(3)}, \gamma^{-1}(a_{(1)}) \rangle \; a_{(2)} \otimes \varphi_{(2)}$$

Lemma 11. *We have the following equalities of linear maps*

$$\begin{aligned}
\tau \circ (\mathrm{id}_B \otimes \mu) \;&= (\mu \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau) \circ (\tau \otimes \mathrm{id}_A) \;:\; B \otimes A \otimes A \to A \otimes B \\
\tau \circ (\Delta^* \otimes \mathrm{id}_A) \;&= (\mathrm{id}_A \otimes \Delta^*) \circ (\tau \otimes \mathrm{id}_B) \circ (\mathrm{id}_B \otimes \tau) \;:\; B \otimes B \otimes A \to A \otimes B
\end{aligned}$$

Proof. Consider the first claimed equation. We take $\varphi \in B$ and $a, b \in A$, and show that the values of both maps on the simple tensor $\varphi \otimes a \otimes b$ are equal. Calculating the left hand side, we use the homomorphism property of coproduct

$$\begin{aligned}
\tau(\varphi \otimes ab) \;&=\; \sum_{(\varphi),(ab)} \langle \varphi_{(1)}, (ab)_{(3)} \rangle \, \langle \varphi_{(3)}, \gamma^{-1}((ab)_{(1)}) \rangle \; (ab)_{(2)} \otimes \varphi_{(2)} \\
&=\; \sum_{(\varphi),(a),(b)} \langle \varphi_{(1)}, a_{(3)} b_{(3)} \rangle \, \langle \varphi_{(3)}, \gamma^{-1}(a_{(1)} b_{(1)}) \rangle \; a_{(2)} b_{(2)} \otimes \varphi_{(2)}.
\end{aligned}$$

We then calculate the right hand side using in the second and third steps coassociativity and definition of the coproduct $\mu^*|_B$ in $B \subset A^\circ$, respectively,

$$\begin{aligned}
&(\mu \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau) \circ (\tau \otimes \mathrm{id}_A)(\varphi \otimes a \otimes b) \\
=\;& (\mu \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau)\Big(\sum_{(\varphi),(a)} \langle \varphi_{(1)}, a_{(3)} \rangle \, \langle \varphi_{(3)}, \gamma^{-1}(a_{(1)}) \rangle \; a_{(2)} \otimes \varphi_{(2)} \otimes b \Big) \\
=\;& (\mu \otimes \mathrm{id}_B)\Big(\sum_{(\varphi),(a),(b)} \langle \varphi_{(1)}, a_{(3)} \rangle \, \langle \varphi_{(5)}, \gamma^{-1}(a_{(1)}) \rangle \, \langle \varphi_{(2)}, b_{(3)} \rangle \, \langle \varphi_{(4)}, \gamma^{-1}(b_{(1)}) \rangle \; a_{(2)} \otimes b_{(2)} \otimes \varphi_{(3)} \Big) \\
=\;& \sum_{(\varphi),(a),(b)} \langle \varphi_{(1)}, a_{(3)} b_{(3)} \rangle \, \langle \varphi_{(3)}, \gamma(b_{(1)}) \gamma^{-1}(a_{(1)}) \rangle \; a_{(2)} b_{(2)} \otimes \varphi_{(2)}.
\end{aligned}$$

By the anti-homomorphism property of γ^{-1}, the expressions are equal, so we have proved the first equality. The proof of the second equality is similar. □

Sketch of a proof of the Hopf algebra axioms in Theorem 9. Let us check associativity (H1) of $\mathcal{D}(A,B)$. Using first Equation (4.18), then the second formula in Lemma 11, and finally changing the order of maps that operate in different components, we calculate

$$\begin{aligned}
\mu_{\mathcal{D}} \circ (\mu_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}}) \;&= (\mu \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B) \circ (\mu \otimes \Delta^* \otimes \mathrm{id}_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B \otimes \mathrm{id}_A \otimes \mathrm{id}_B) \\
&= (\mu \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \Delta^* \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \mathrm{id}_B \otimes \tau \otimes \mathrm{id}_B) \\
&\qquad \circ (\mu \otimes \mathrm{id}_B \otimes \mathrm{id}_B \otimes \mathrm{id}_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B \otimes \mathrm{id}_A \otimes \mathrm{id}_B) \\
&= (\mu \otimes \Delta^*) \circ ((\mu \otimes \mathrm{id}_A) \otimes (\Delta^* \otimes \mathrm{id}_B)) \circ (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau \otimes \tau \otimes \mathrm{id}_B).
\end{aligned}$$

Likewise, with the first of the formulas in the lemma, we calculate

$$\begin{aligned}
\mu_{\mathcal{D}} \circ (\mathrm{id}_{\mathcal{D}} \otimes \mu_{\mathcal{D}}) \;&= (\mu \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \mathrm{id}_B \mu \otimes \mu \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \mathrm{id}_B \otimes \mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B) \\
&= (\mu \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \mu \otimes \mathrm{id}_B \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A \otimes \mathrm{id}_B \otimes \mathrm{id}_B) \\
&\qquad \circ (\mathrm{id}_A \otimes \mathrm{id}_B \otimes \mathrm{id}_A \otimes \mathrm{id}_A \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \mathrm{id}_B \otimes \mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B) \\
&= (\mu \otimes \Delta^*) \circ ((\mathrm{id}_A \otimes \mu) \otimes (\mathrm{id}_B \otimes \Delta^*)) \circ (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau \otimes \tau \otimes \mathrm{id}_B).
\end{aligned}$$

Using associativity (H1) for both algebras (A, μ, η) and $(B, \Delta^*, \epsilon^*)$ we see that these two expressions match and associativity follows for the Drinfeld double $\mathcal{D}(A,B)$.

Some of the other axioms are very easy to check. Consider for example coassociativity (H1’) of $\mathcal{D}(A,B)$. In view of Equation (4.15), and coassociativity of both A and B, we have

$$\begin{aligned}
(\Delta_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}}) \circ \Delta_{\mathcal{D}}(a \otimes \varphi) \;&= \sum_{(a)} \sum_{(\varphi)} \sum_{(a_{(1)})} \sum_{(\varphi_{(2)})} (a_{(1)})_{(1)} \otimes (\varphi_{(2)})_{(2)} \otimes (a_{(1)})_{(2)} \otimes (\varphi_{(2)})_{(1)} \otimes a_{(2)} \otimes \varphi_{(1)} \\
&= \sum_{(a)} \sum_{(\varphi)} \sum_{(a_{(2)})} \sum_{(\varphi_{(1)})} a_{(1)} \otimes \varphi_{(2)} \otimes (a_{(2)})_{(1)} \otimes (\varphi_{(1)})_{(2)} \otimes (a_{(2)})_{(2)} \otimes (\varphi_{(1)})_{(1)} \\
&= (\mathrm{id}_{\mathcal{D}} \otimes \Delta_{\mathcal{D}}) \circ \Delta_{\mathcal{D}}(a \otimes \varphi).
\end{aligned}$$

□

A Drinfeld double of H_q and the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$

A Drinfeld double of H_q for q not a root of unity

Let $q \in \mathbb{C} \setminus \{0\}$, and assume throughout that $q^N \neq 1$ for all $N \neq 0$. Recall that as an algebra, H_q is generated by elements a, a^{-1}, b subject to the relations

$$a\,a^{-1} = 1 \quad , \qquad a^{-1}\,a = 1 \quad , \qquad a\,b = q\;b\,a.$$

The Hopf algebra structure on H_q is then uniquely determined by the coproducts of a and b,

$$\Delta(a) = a \otimes a \quad , \qquad \Delta(b) = a \otimes b + b \otimes 1.$$

We have considered the elements $1^*, \tilde{a}, \tilde{a}^{-1}, \tilde{b} \in H_q^\circ$ given by

$$\langle 1^*, b^m a^n \rangle \; = \; \delta_{m,0} \quad , \qquad \langle \tilde{a}^{\pm 1}, b^m a^n \rangle \; = \; \delta_{m,0}\, q^{\pm n} \quad , \qquad \langle \tilde{b}, b^m a^n \rangle \; = \; \delta_{m,1}.$$

Let $H_q' \subset H_q^\circ$ be the Hopf subalgebra of H_q° generated by these elements. By Lemma 1, H_q' is isomorphic to H_q as a Hopf algebra by the isomorphism which sends $a \mapsto \tilde{a}$ and $b \mapsto \tilde{b}$. In particular, $(\tilde{b}^m\, \tilde{a}^n)_{m \in \mathbb{N}, n \in \mathbb{Z}}$ is a basis of H_q'.

For the Drinfeld double we need the inverse of the antipode. This is given in the following.

Exercise 5. *In H_q, the antipode γ is invertible and its inverse is given by*

$$\gamma^{-1}(b^m a^n) \; = \; (-1)^m \, q^{-\frac{1}{2}m(m-1)-mn} \, b^m a^{-m-n}.$$

Therefore we can consider the associated Drinfeld double, $D_q = \mathcal{D}(H_q, H_q')$. Both H_q and H_q' are embedded in D_q, so let us choose the following notation for the embedded generators

$$\alpha \; = \; \iota_{H_q}(a) = a \otimes 1^* \qquad \beta \; = \; \iota_{H_q}(b) \; = \; b \otimes 1^* \qquad \tilde{\alpha} \; = \; \iota_{H_q'}(\tilde{a}) = 1 \otimes \tilde{a} \qquad \tilde{\beta} \; = \; \iota_{H_q'}(\tilde{b}) = 1 \otimes \tilde{b}.$$

We have, by properties (i), (ii), (iii) of Drinfeld double

$$\beta^m\, \alpha^n\, \tilde{\beta}^{m'}\, \tilde{\alpha}^{n'} \; = \; b^m a^n \otimes \tilde{b}^{m'}\, \tilde{a}^{n'}$$

and these elements, for $m, m' \in \mathbb{N}$ and $n, n' \in \mathbb{Z}$ form a basis of D_q.

Let us start by calculating the products of the elements $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in D_q$. Among the products of the generators of D_q, property (i) makes those involving only α and β trivial, and property (ii) makes those involving only $\tilde{\alpha}$ and $\tilde{\beta}$ trivial. Also by property (iii) there is nothing to calculate for the products $\alpha\, \tilde{\alpha}$, $\alpha\, \tilde{\beta}$, $\beta\, \tilde{\alpha}$, $\beta\, \tilde{\beta}$. For the rest, we need the double coproducts of a and b,

$$\begin{aligned}
(\Delta \otimes \mathrm{id}_{H_q}) \circ \Delta(a) \; &= a \otimes a \otimes a \\
(\Delta \otimes \mathrm{id}_{H_q}) \circ \Delta(b) \; &= a \otimes a \otimes b + a \otimes b \otimes 1 + b \otimes 1 \otimes 1,
\end{aligned}$$

and of $\tilde{a}$ and $\tilde{b}$ for which the formulas are the same. We also need particular cases of Exercise 5:

$$\gamma^{-1}(a) \; = \; a^{-1} \quad , \qquad \gamma^{-1}(b) \; = \; -b\, a^{-1}.$$

The products that require short calculations are

$$\tilde{\alpha}\, \alpha \; = \; \underbrace{\langle \tilde{a}, a \rangle}_{=q} \; \underbrace{\langle \tilde{a}, \gamma^{-1}(a) \rangle}_{=q^{-1}} \; a \otimes \tilde{a} \; = \; \alpha\, \tilde{\alpha}$$

and

$$\begin{aligned}
\tilde{\alpha}\, \beta \; = \; & \underbrace{\langle \tilde{a}, b \rangle}_{=0} \; \underbrace{\langle \tilde{a}, \gamma^{-1}(a) \rangle}_{=q^{-1}} \; a \otimes \tilde{a} + \underbrace{\langle \tilde{a}, 1 \rangle}_{=1} \; \underbrace{\langle \tilde{a}, \gamma^{-1}(a) \rangle}_{=q^{-1}} \; b \otimes \tilde{a} + \underbrace{\langle \tilde{a}, 1 \rangle}_{=1} \; \underbrace{\langle \tilde{a}, \gamma^{-1}(b) \rangle}_{=0} \; 1 \otimes \tilde{a} \\
= \; & q^{-1}\, \beta\, \tilde{\alpha}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\beta}\, \alpha \; = \; & \underbrace{\langle \tilde{a}, a \rangle}_{=q} \; \underbrace{\langle \tilde{b}, \gamma^{-1}(a) \rangle}_{=0} \; a \otimes \tilde{a} + \underbrace{\langle \tilde{a}, a \rangle}_{=q} \; \underbrace{\langle 1^*, \gamma^{-1}(a) \rangle}_{=1} \; a \otimes \tilde{a} + \underbrace{\langle \tilde{b}, a \rangle}_{=0} \; \underbrace{\langle 1^*, \gamma^{-1}(a) \rangle}_{=1} \; a \otimes \tilde{a} \\
= \; & q\, \alpha\, \tilde{\beta}
\end{aligned}$$

and

$$\begin{aligned}\tilde{\beta}\,\beta \;=\; & \langle\tilde{a},b\rangle\,\langle\tilde{b},\gamma^{-1}(a)\rangle\,a\otimes\tilde{a}+\langle\tilde{a},1\rangle\,\langle\tilde{b},\gamma^{-1}(a)\rangle\,b\otimes\tilde{a}+\langle\tilde{a},1\rangle\,\langle\tilde{b},\gamma^{-1}(b)\rangle\,1\otimes\tilde{a}\\
&+\langle\tilde{a},b\rangle\,\langle 1^*,\gamma^{-1}(a)\rangle\,a\otimes\tilde{b}+\langle\tilde{a},1\rangle\,\langle 1^*,\gamma^{-1}(a)\rangle\,b\otimes\tilde{b}+\langle\tilde{a},1\rangle\,\langle 1^*,\gamma^{-1}(b)\rangle\,1\otimes\tilde{b}\\
&+\langle\tilde{b},b\rangle\,\langle 1^*,\gamma^{-1}(a)\rangle\,a\otimes 1^*+\langle\tilde{b},1\rangle\,\langle 1^*,\gamma^{-1}(a)\rangle\,b\otimes 1^*+\langle\tilde{b},1\rangle\,\langle 1^*,\gamma^{-1}(b)\rangle\,1\otimes 1^*\\
=\; & -\tilde{\alpha}+\beta\,\tilde{\beta}+\alpha.\end{aligned}$$

We get the following description of D_q.

Proposition 12. *The Hopf algebra D_q is, as an algebra, generated by elements $\alpha,\alpha^{-1},\beta,\tilde{\alpha},\tilde{\alpha}^{-1},\tilde{\beta}$ with relations*

$$\begin{aligned}\alpha\alpha^{-1} &= 1 = \alpha^{-1}\alpha & \tilde{\alpha}\tilde{\alpha}^{-1} &= 1 = \tilde{\alpha}^{-1}\tilde{\alpha}\\
\alpha\beta &= q\,\beta\alpha & \tilde{\alpha}\tilde{\beta} &= q\,\tilde{\beta}\tilde{\alpha}\\
\alpha\tilde{\beta} &= q^{-1}\,\tilde{\beta}\alpha & \tilde{\alpha}\beta &= q^{-1}\,\beta\tilde{\alpha}\\
\alpha\tilde{\alpha} &= \tilde{\alpha}\alpha & \tilde{\beta}\beta-\beta\tilde{\beta} &= \alpha-\tilde{\alpha}.\end{aligned}$$

The Hopf algebra structure on D_q is the unique one such that

$$\Delta(\alpha) = \alpha\otimes\alpha \qquad \Delta(\tilde{\alpha}) = \tilde{\alpha}\otimes\tilde{\alpha} \qquad \Delta(\beta) = \alpha\otimes\beta+\beta\otimes 1 \qquad \Delta(\tilde{\beta}) = \tilde{\beta}\otimes\tilde{\alpha}+1\otimes\tilde{\beta}.$$

Proof. It is clear that the elements generate D_q, and we have just shown that the above relations hold for the generators. Using the relations it is possible to express any element of D_q as a linear combination of the vectors $\beta^m\alpha^n\tilde{\beta}^{m'}\tilde{\alpha}^{n'}$. Since these are linearly independent in D_q, it follows that the algebra D_q has a presentation given by the generators and relations as stated. The coproduct formulas for α, $\tilde{\alpha}$, β, $\tilde{\beta}$ are obvious in view of requirements (i) and (ii) of Drinfeld double, and it is a standard calculation to show that the structural maps are determined by the given values. □

The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ as a quotient of D_{q^2}

To take quotients of Hopf algebras we need the notion of Hopf ideals. A vector subspace J in a Hopf algebra H is a Hopf ideal if J is a two-sided ideal of H as an algebra (i.e. $\mu(J\otimes H)\subset J$ and $\mu(H\otimes J)\subset J$), and J is a coideal of H as a coalgebra (i.e. $\Delta(J)\subset J\otimes H+H\otimes J$ and $\epsilon|_J = 0$) and J is an invariant subspace for the antipode (i.e. $\gamma(J)\subset J$). These requirements are precisely what one needs for the structural maps to be well defined on the equivalence classes $x+J$ that form the quotient space H/J.

Lemma 13. *The element $\kappa = \alpha\,\tilde{\alpha}$ is a grouplike central element in D_q, and the two-sided ideal J_q generated by $\kappa-1$ is a Hopf ideal.*

Proof. We have

$$\Delta(\kappa) = \Delta(\alpha\tilde{\alpha}) = \Delta(\alpha)\,\Delta(\tilde{\alpha}) = (\alpha\otimes\alpha)\,(\tilde{\alpha}\otimes\tilde{\alpha}) = (\alpha\tilde{\alpha}\otimes\alpha\tilde{\alpha}) = \kappa\otimes\kappa,$$

so κ is grouplike. To show that it is central, it suffices to show that it commutes with the generators, but this is easily seen from the relations in Proposition 12: for example

$$\begin{aligned}\alpha\kappa &= \alpha\alpha\tilde{\alpha} = \alpha\tilde{\alpha}\alpha = \kappa\alpha\\
\beta\kappa &= \beta\alpha\tilde{\alpha} = q^{-1}\,\alpha\beta\tilde{\alpha} = q^{-1}\,q\,\alpha\tilde{\alpha}\beta = \kappa\beta,\end{aligned}$$

and similarly for commutation with $\tilde{\alpha}$ and $\tilde{\beta}$. The two sided ideal generated by $\kappa-1$ is spanned by elements of the form $x(\kappa-1)y$, where $x,y\in D_q$. To show that it is a coideal, we first compute

$$\Delta(\kappa-1) = \kappa\otimes\kappa-1\otimes 1 = (\kappa-1)\otimes\kappa+1\otimes(\kappa-1) \in J_q\otimes D_q+D_q\otimes J_q.$$

Then, using $\Delta(x(\kappa - 1)y) = \Delta(x)\,\Delta(\kappa - 1)\,\Delta(y)$, the more general result $\Delta(J_q) \subset J_q \otimes D_q + D_q \otimes J_q$ follows. To show that J_q is stable under antipode, we first compute

$$\gamma(\kappa - 1) \;=\; \tilde{\alpha}^{-1}\alpha^{-1} - 1 \;=\; \tilde{\alpha}^{-1}\alpha^{-1}(1 - \alpha\tilde{\alpha}) \;=\; -\tilde{\alpha}^{-1}\alpha^{-1}(\kappa - 1) \;\in\; J_q.$$

Then, using $\gamma(x(\kappa-1)y) = \gamma(y)\,\gamma(\kappa-1)\,\gamma(x)$, the more general result $\gamma(J_q) \subset J_q$ follows. To show that $\epsilon|_{J_q} = 0$ note that $\epsilon(\kappa-1) = \epsilon(\kappa)-\epsilon(1) = 1-1 = 0$ and thus also $\epsilon(x(\kappa-1)y) = \epsilon(x)\,\epsilon(\kappa-1)\,\epsilon(y) = 0$. □

We can now take the quotient Hopf algebra D_q/J_q. Let us summarize what we have done, then. We've taken two copies of the building block, or the "quantum Borel subalgebra" H_q and put them together by the Drinfeld double construction as $D_q = \mathcal{D}(H_q, H_q')$ — one of the copies has generators α and β, and the other has generators $\tilde{\alpha}$ and $\tilde{\beta}$. Then we have identified their "quantum Cartan subalgebras", generated respectively by α and $\tilde{\alpha}$, by requiring $\alpha = \tilde{\alpha}^{-1}$ (which is equivalent to $\kappa - 1 = 0$). This is a way to obtain essentially $\mathcal{U}_q(\mathfrak{sl}_2)$, although, to be consistent with common usage, we redefine the parameter q and use q^2 instead.

If we use the notations K, E and F for the equivalence classes in D_{q^2}/J_{q^2} of $\tilde{\alpha}$, $\frac{-1}{q-q^{-1}}\,\tilde{\beta}$ and β, respectively, then the relations in Proposition 12 become the ones in the following definition of $\mathcal{U}_q(\mathfrak{sl}_2)$.

Definition 3. *Let $q \in \mathbb{C} \setminus \{0, +1, -1\}$. The algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is the algebra generated by elements E, F, K, K^{-1} with relations*

$$K\,K^{-1} = 1 = K^{-1}\,K \qquad\qquad K\,E\,K^{-1} = q^2\,E$$
$$E\,F - F\,E = \frac{1}{q - q^{-1}}\left(K - K^{-1}\right) \qquad\qquad K\,F\,K^{-1} = q^{-2}\,F.$$

We equip $\mathcal{U}_q(\mathfrak{sl}_2)$ with the unique Hopf algebra structure such that

$$\Delta(K) \;=\; K \otimes K \quad , \qquad \Delta(E) \;=\; E \otimes K + 1 \otimes E \quad , \qquad \Delta(E) \;=\; K^{-1} \otimes F + F \otimes 1.$$

An easy comparison of the above definition with Proposition 12 and Lemma 13 gives the following.

Proposition 14. *When q is not a root of unity, then the Hopf algebras $\mathcal{U}_q(\mathfrak{sl}_2)$ and D_{q^2}/J_{q^2} are isomorphic.*

A convenient Poincaré-Birkhoff-Witt type basis of $\mathcal{U}_q(\mathfrak{sl}_2)$ is

$$(F^m\,K^k\,E^n)_{m,n\in\mathbb{N},k\in\mathbb{Z}}.$$

For working with the above parametrization, with q^2 replacing what used to be q, it is convenient to use the following more symmetric q-integers and q-factorials, which we denote as

$$[n] \;=\; \frac{q^n - q^{-n}}{q - q^{-1}} \tag{4.19}$$
$$[n]! \;=\; [n]\,[n-1]\,\cdots\,[2]\,[1] \tag{4.20}$$
$$\begin{bmatrix} n \\ k \end{bmatrix} \;=\; \frac{[n]!}{[k]!\;[n-k]!} \tag{4.21}$$

when considered as rational functions of q, and as

$$[n]_q \quad , \qquad [n]_q! \quad , \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q \quad ,$$

respectively, when evaluated at a value $q \in \mathbb{C} \setminus \{0\}$.

Exercise 6. *Show the following properties of the q-integers, q-factorials and q-binomials* *(a)* $[n] = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$ *and* $[n]_q = q^{1-n}\, [\![n]\!]_{q^2}$

(b) $[m+n] = q^n\,[m] + q^{-m}\,[n] = q^{-n}\,[m] + q^m\,[n]$

(c) $[l]\,[m-n] + [m]\,[n-l] + [n]\,[l-m] = 0$

(d) $[n] = [2]\,[n-1] - [n-2]$.

Representations of D_{q^2} and $\mathcal{U}_q(\mathfrak{sl}_2)$

Let us now start analyzing representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ and the closely related Hopf algebra D_{q^2}. The general story is very much parallel with the (more familiar) case of representations of $\mathfrak{sl}_2$. In particular, in a given $\mathcal{U}_q(\mathfrak{sl}_2)$-module V we will attempt to diagonalize K, and then notice that if v is an eigenvector of K with eigenvalue λ,

$$K.v = \lambda\, v,$$

then $E.v$ and $F.v$ also either vanish or are eigenvectors of eigenvalues $q^{\pm 2}\lambda$,

$$K.(E.v) = KE.v = q^2\, EK.v = q^2\lambda\, E.v \quad , \qquad K.(F.v) = KF.v = q^{-2}\, FK.v = q^{-2}\lambda\, F.v.$$

The situation is nicest if q^2 is not a root of unity, so that repeated application of E (or F) on an eigenvector produces other eigenvectors with distinct eigenvalues.

Another useful observation for studying representations is the following, very much analoguous to the quadratic Casimir element of ordinary $\mathfrak{sl}_2$.

Lemma 15. *The elements $C \in \mathcal{U}_q(\mathfrak{sl}_2)$ and $\nu \in D_{q^2}$ given by*

$$\begin{aligned} C &= E\,F + \frac{1}{(q-q^{-1})^2}\Big(q^{-1}\,K + q\,K^{-1}\Big) \\ &= F\,E + \frac{1}{(q-q^{-1})^2}\Big(q\,K + q^{-1}\,K^{-1}\Big) \end{aligned}$$

and

$$\begin{aligned} \nu &= \tilde{\beta}\,\beta + \frac{q}{q-q^{-1}}\,\alpha + \frac{q^{-1}}{q-q^{-1}}\,\tilde{\alpha} \\ &= \beta\,\tilde{\beta} + \frac{q^{-1}}{q-q^{-1}}\,\alpha + \frac{q}{q-q^{-1}}\,\tilde{\alpha} \end{aligned}$$

are central.

Proof. Let us first show that the two formulas for C are equal. Their difference is

$$E\,F - F\,E + \frac{1}{(q-q^{-1})^2}\Big((q^{-1}-q)K + (q-q^{-1})K^{-1}\Big).$$

After canceling one factor $q - q^{-1}$ from the numerator and denominator, this is seen to be zero by one of the defining relations of $\mathcal{U}_q(\mathfrak{sl}_2)$.

To show that C is central, it suffices to show that it commutes with the generators K, E and F. Commutation with K is evident, since $KEF = q^2\,EKF = EFK$ and the second term of C is a polynomial in K and K^{-1}. To show commutation with E, calculate CE using the first expression for C to get

$$C\,E \;=\; E\,F\,E + \frac{1}{(q-q^{-1})^2}\Big(q^{-1}\,K\,E + q\,K^{-1}\,E\Big)$$

and EC using the second expression for C to get

$$E\,C \;=\; E\,F\,E + \frac{1}{(q-q^{-1})^2}\Big(q\,E\,K + q^{-1}\,E\,K^{-1}\Big).$$

Then it suffices to recall the relations $KE = q^2\,EK$ and $K^{-1}E = q^{-2}\,EK^{-1}$ to see the equality $CE = EC$. The commutation of C with F is shown similarly.

The verification that ν is central in D_{q^2} is left as an exercise. For q not a root of unity, the first statement in fact follows from the second by passing to the quotient $\mathcal{U}_q(\mathfrak{sl}_2) \cong D_{q^2}/J_{q^2}$. □

On representations of D_{q^2}

We will start by analyzing representations of D_{q^2}, because every representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ can be interpreted as a representation of D_{q^2}, where $\kappa = \alpha\tilde{\alpha}$ acts as identity. Note that we thus assume q is not a root of unity, so that D_{q^2} is defined and $\mathcal{U}_q(\mathfrak{sl}_2) \cong D_{q^2}/J_{q^2}$. The case when q is a root of unity is more complicated in terms of representation theory and has to be treated separately anyway.

We will first look for irreducible representations of D_{q^2}, i.e. simple D_{q^2}-modules. Note first of all the following general principle (essentially the same as Schur's lemma).

Lemma 16. *If V is a finite dimensional irreducible representation of an algebra A, and if $c \in A$ is a central element, then there is a $\lambda \in \mathbb{C}$ such that c acts as $\lambda\,\mathrm{id}_V$ on V.*

Proof. It is always possible to find one eigenvector of c, with eigenvalue that is a root of the characteristic polynomial. Call the eigenvalue λ and note that $c - \lambda\,\mathrm{id}_V$ is a D_{q^2}-module map $V \to V$ with a nontrivial kernel. The kernel is a subrepresentation, so by irreducibility it has to be the whole V. □

Because of the above principle, we will in what follows consider only representations of D_{q^2} where $\kappa = \alpha\tilde{\alpha}$ acts as λ id. As a consequence $\tilde{\alpha}$ has the same action as $\lambda\,\alpha^{-1}$.

Suppose now that V is an irreducible representation of D_{q^2}, and denote the (only) eigenvalue of κ by $\lambda \neq 0$. Take an eigenvector v of α, so $\alpha.v = \mu'\,v$ for some $\mu' \neq 0$. Now an easy computation shows that the vectors $\tilde{\beta}^j.v$ are either eigenvectors of α with eigenvalue $q^{-2j}\mu'$, or zero vectors. Since these eigenvalues are different and eigenvectors corresponding to different eigenvalues are linearly independent, we see that if V is finite dimensional, then there must be a $j > 0$ such that the vector $w_0 = \tilde{\beta}^{j-1}.v$ satisfies

$$\tilde{\beta}.w_0 \;=\; 0 \qquad \text{and} \qquad \alpha.w_0 \;=\; \mu\,w_0,$$

where $\mu = q^{2(1-j)}\mu'$. Denote $w_j = \beta^j.w_0$. Again, w_j are eigenvectors of α with eigenvalues $q^{2j}\mu$, so for some $d \in \mathbb{N}$ we have

$$w_{d-1} \;=\; \beta^{d-1}.w_0 \;\neq\; 0 \qquad \text{but} \qquad w_d \;=\; \beta.w_{d-1} \;=\; \beta^d.w_0 \;=\; 0.$$

We claim that the linear span $W \subset V$ of $\{w_0, w_1, w_2, \ldots, w_{d-1}\}$ is a subrepresentation, and thus by irreducibility $W = V$. We have

$$\alpha.w_j \;=\; q^{2j}\mu\,w_j \qquad \text{and} \qquad \tilde{\alpha}.w_j \;=\; q^{-2j}\frac{\lambda}{\mu}\,w_j,$$

so W is stable under the action of $\alpha, \tilde{\alpha}$ and β. We must only verify that $\tilde{\beta}$ preserves W. Calculate the action of $\tilde{\beta}$ on w_j commuting $\tilde{\beta}$ to the right of all β, and finally recalling that $\tilde{\beta}.w_0 = 0$,

$$\begin{aligned}
\tilde{\beta}.w_j &= \tilde{\beta}\beta^j.w_0 \;=\; (\beta\tilde{\beta} + \alpha - \tilde{\alpha})\beta^{j-1}.w_0 \\
&= \beta\tilde{\beta}\beta^{j-1}.w_0 + (q^{2(j-1)}\mu - q^{-2(j-1)}\frac{\lambda}{\mu})\,\beta^{j-1}.w_0 \\
&= \beta(\beta\tilde{\beta} + \alpha - \tilde{\alpha})\beta^{j-2}.w_0 + \left(q^{2(j-1)}\mu - q^{-2(j-1)}\frac{\lambda}{\mu}\right)\beta^{j-1}.w_0 \\
&= \beta^2\tilde{\beta}\beta^{j-2}.w_0 + \left((q^{2(j-1)} + q^{2(j-2)})\mu - (q^{-2(j-1)} + q^{-2(j-2)})\frac{\lambda}{\mu}\right)\beta^{j-1}.w_0 \\
&= \cdots \\
&= \beta^j\tilde{\beta}.w_0 + \left((q^{2(j-1)} + q^{2(j-2)} + \cdots + q^2 + 1)\mu - (q^{-2(j-1)} + q^{-2(j-2)} + \cdots + q^{-2} + 1)\frac{\lambda}{\mu}\right)\beta^{j-1}.w_0 \\
&= [j]_q \left(q^{j-1}\mu - q^{1-j}\frac{\lambda}{\mu}\right) w_{j-1}.
\end{aligned}$$

This finishes the proof that W is a subrepresentation. We will finally obtain a relation between the values of μ, λ and d. For this, note that $\beta^d.w_0 = w_d = 0$. Thus also $\tilde{\beta}\beta^d.w_0 = 0$. But the above calculation is still valid and it says that $\tilde{\beta}\beta^d.w_0$ is a constant multiple of w_{d-1}, with the constant $[d]_q\,(q^{d-1}\mu - q^{1-d}\lambda/\mu)$. This constant must therefore vanish, ans since the q-integers are non-zero, we get the following relation between the parameters λ, μ and d

$$\mu^2 \;=\; q^{2(1-d)}\lambda. \tag{4.22}$$

Given $\lambda \in \mathbb{C} \setminus \{0\}$ and $d \in \mathbb{N}$, the two solutions for μ are

$$\mu \;=\; \pm q^{1-d}\sqrt{\lambda}.$$

In particular, the eigenvalues of α on W are of the form $q^{2j}\mu$ and those of $\tilde{\alpha}$ are $q^{-2j}\lambda/\mu$, so the spectra of both consist of

$$\pm\sqrt{\lambda}q^{1-d},\; \pm\sqrt{\lambda}q^{3-d},\; \ldots,\; \pm\sqrt{\lambda}q^{d-3},\; \pm\sqrt{\lambda}q^{d-1}.$$

Note also that the action of $\tilde{\beta}$ simplifies a bit,

$$\tilde{\beta}.w_j \;=\; \pm\sqrt{\lambda}\,(q^{-1} - q)\,[j]_q\,[d-j]_q\,w_{j-1}.$$

We have in fact found all the irreducible finite dimensional representations of D_{q^2}.

Theorem 17. *For any nonzero complex number λ and a choice of square root $\sqrt{\lambda}$, and d a positive integer, there exists a d-dimensional irreducible representation $W_d^{(\sqrt{\lambda})}$ of D_{q^2} with basis $\{w_0, w_1, w_2, \ldots, w_{d-1}\}$ such that*

$$\begin{aligned}
\alpha.w_j &= \sqrt{\lambda}\,q^{1-d+2j}\,w_j & \qquad \tilde{\alpha}.w_j &= \sqrt{\lambda}\,q^{d-1-2j}\,w_j \\
\beta.w_j &= w_{j+1} & \tilde{\beta}.w_j &= \sqrt{\lambda}\,[j]_q\,[d-j]_q\,(q^{-1} - q)\,w_{j-1}.
\end{aligned}$$

Any finite dimensional D_{q^2}-module contains a submodule isomorphic to some $W_d^{(\sqrt{\lambda})}$, and in particular there are no other finite dimensional irreducible D_{q^2} modules.

Proof. To verify that the formulas indeed define a representation is straightforward and the calculations are essentially the same as above. To verify irreducibility of $W_d^{(\sqrt{\lambda})}$, note that if $W' \subset W_d^{(\sqrt{\lambda})}$ is a non-zero submodule, then it contains some eigenvector of α, which must be proportional to some w_j. Then by the repeated action of $\tilde{\beta}$ and β we see that W' contains all w_j, $j = 0, 1, 2, \ldots, d-1$ (note that the coefficient $\sqrt{\lambda}\,[j]_q\,[d-j]_q\,(q^{-1} - q)$ is never zero for $j = 1, 2, \ldots, d-1$). Above we already showed that any finite dimensional D_{q^2} module V must contain a submodule isomorphic to $W_d^{(\pm\sqrt{\lambda})}$, so it follows indeed that these are all the possible irreducible D_{q^2}-modules. $\square$

Since any representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ is a representation of D_{q^2} such that $\lambda = 1$, we have also found all irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_2)$. To get the explicit formulas, recall that the generators K, E, F correspond to the equivalence classes of $\tilde{\alpha}$, $\frac{-1}{q-q^{-1}}\tilde{\beta}$ and β modulo the Hopf ideal J_{q^2} generated by the element $\kappa - 1$.

Theorem 18. *Let q be a non-zero complex number which is not a root of unity. For any positive integer d and for $\varepsilon \in \{+1, -1\}$, there exists a d-dimensional irreducible representation W_d^ε of $\mathcal{U}_q(\mathfrak{sl}_2)$ with basis $\{w_0, w_1, w_2, \ldots, w_{d-1}\}$ such that*

$$\begin{aligned} K.w_j &= \varepsilon\, q^{d-1-2j}\, w_j \\ F.w_j &= w_{j+1} \\ E.w_j &= \varepsilon\, [j]_q\, [d-j]_q\, w_{j-1}. \end{aligned}$$

There are no other finite dimensional irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-modules.

Proof. Follows directly from Theorem 17. □

Using the formulas in Lemma 15 one computes that on W_d^ε

$$\text{the central element } C \text{ acts as } \; \varepsilon\, \frac{q^d + q^{-d}}{(q - q^{-1})^2}\, \mathrm{id}_{W_d^\varepsilon}. \tag{4.23}$$

Since the numbers $\pm(q^d + q^{-d})$ are distinct, we see first of all that none of the W_d^ε are isomorphic with each other (of course for different dimension d they couldn't be isomorphic anyway). Thus the value of C distinguishes the different irreducible representations.

On semisimplicity

Definition 4. *Let A be an algebra. An A-module W is called simple (or irreducible) if the only submodules of W are $\{0\}$ and W . An A-module V is called completely reducible if V is isomorphic to a direct sum of finitely many simple A-modules. An algebra A is called semisimple if all finite dimensional A-modules are completely reducible.*

The terms "simple module" and "irreducible representation" seem standard, but we will also speak of irreducible modules with the same meaning.

Definition 5. *An A-module V is called indecomposable if it can not be written as a direct sum of two nonzero submodules.*

In particular any irreducible module is indecomposable. And for semisimple algebras the two concepts are the same.

The following classical result gives equivalent conditions for semisimplicity, which are often practical.

Proposition 19. *Let A be an algebra. The following conditions are equivalent.*

(i) Any finite dimensional A-module is isomorphic to a finite direct sum of irreducible A-modules (i.e. A is semisimple).

(ii) For any finite dimensional A-module V and any submodule $W \subset V$ there exists a submodule $W' \subset V$ (complementary submodule) such that $V = W \oplus W'$ as an A-module.

(iii) For any finite dimensional A-module V and any irreducible submodule $W \subset V$ there exists a submodule $W' \subset V$ such that $V = W \oplus W'$ as an A-module.

(iv) *For any finite dimensional A-module V and any submodule $W \subset V$ there exists an A-module map $\pi : V \to W$ such that $\pi|_W = \mathrm{id}_W$ (an A-linear projection to the submodule).*

(v) *For any finite dimensional A-module V and any irreducible submodule $W \subset V$ there exists an A-module map $\pi : V \to W$ such that $\pi|_W = \mathrm{id}_W$.*

Proof. Clearly *(ii)* $\Rightarrow$ *(iii)* and *(iv)* $\Rightarrow$ *(v)*.

Let us show that *(ii)* and *(iv)* are equivalent, in the same way one shows that *(iii)* and *(v)* are equivalent. Assume *(ii)*, that any submodule $W \subset V$ has a complementary submodule W', that is $V = W \oplus W'$. Then if π is the projection to W with respect to this direct sum decomposition, we have that for all $w \in W$, $w' \in W'$, $a \in A$

$$\pi\left(a \cdot (w + w')\right) \;=\; \pi(a \cdot w + a \cdot w') \;=\; a \cdot w \;=\; a \cdot \pi(w + w'),$$

which shows that the projection is A-linear. Conversely, assume *(iv)* that for any submodule $W \subset V$ there is an A-linear projection $\pi : V \to W$. The subspace $W' = \mathrm{Ker}\,(\pi)$ is a submodule complementary to $W = \mathrm{Ker}\,(1 - \pi)$.

We must still show for example that *(iii)* $\Rightarrow$ *(i)* and *(i)* $\Rightarrow$ *(ii)*.

Assume *(iii)* and let V be a finite dimensional A-module (we may assume immediately that $V \neq \{0\}$). Consider a non-zero submodule $W_1 \subset V$ of smallest dimension, it is necessarily irreducible. If $W_1 = V$ we're done, if not by property *(iii)* we have a complementary submodule $V_1 \subset V$ with $\dim\, V_1 < \dim\, V$ and $V = W_1 \oplus V_1$. Continue recursively to find the non-zero irreducible submodules $W_n \subset V_{n-1}$ and their complementaries V_n in V_{n-1}, that is $V_{n-1} = W_n \oplus V_n$. The dimensions of the latter are strictly decreasing,

$$\dim\, V > \dim\, V_1 > \dim\, V_2 > \cdots,$$

so for some $n_0 \in \mathbb{N}$ we have $W_n = V_{n-1}$ and

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_{n_0},$$

proving *(i)*.

Let us finally prove that *(i)* implies *(ii)*. Suppose $V = \bigoplus_{i \in I} W_i$, where I is a finite index set and for all $i \in I$ the submodule W_i is irreducible. Suppose $W \subset V$ is a submodule, and choose a subset $J \subset I$ such that

$$W \cap \left(\bigoplus_{j \in J} W_j \right) \;=\; \{0\}\,, \tag{4.24}$$

but that for all $i \in I \setminus J$

$$W \cap \left(W_i \oplus \bigoplus_{j \in J} W_j \right) \;\neq\; \{0\}\,. \tag{4.25}$$

Denote by $W' = \bigoplus_{j \in J} W_j$ the submodule thus obtained. By Equation (4.24) the sum of W and W' is direct, and we will prove that it is the entire module V. For that, note that by Equation (4.25) for all $i \in I \setminus J$ there exists $w \in W \setminus \{0\}$ such that $w = w_i + w'$ with $w_i \in W_i \setminus \{0\}$, $w' \in W'$. Therefore the submodule $W \oplus W'$ contains the nonzero vector $w_i \in W_i$, and by irreducibility we get $W_i \subset W \oplus W'$. We get this inclusion for all $i \in I \setminus J$, and also evidently $W_j \subset W \oplus W'$ for all $j \in J$, so we conclude

$$V \;=\; \bigoplus_{i \in I} W_i \;\subset\; W \oplus W',$$

which finishes the proof. $\square$

We will verify complete reducibility of $\mathcal{U}_q(\mathfrak{sl}_2)$ (for q not a root of unity) using the following criterion.

Proposition 20. *Suppose that A is a Hopf algebra for which the following criterion holds:*

- *Whenever R is an A-module and $R_0 \subset R$ is a submodule such that R/R_0 is isomorphic to the one-dimensional trivial A-module, then R_0 has a complementary submodule P (which then must be one-dimensional and trivial).*

Then A is semisimple.

Remark 3. *Actually the criterion can be stated in a superficially weaker form: it suffices that whenever R is an A-module and $R_0 \subset R$ is an* irreducible *submodule of codimension one such that R/R_0 is a trivial module, then there is a complementary submodule P to R_0. Indeed, assuming this weaker condition we can perform an induction on dimension to get to the general case. If R_0 is not irreducible, take a nontrivial irreducible submodule $S_0 \subset R_0$. Then consider the module R/S_0 and its submodule R_0/S_0 of codimension one, which is trivial since $(R/S_0)/(R_0/S_0) \cong R/R_0$. The dimensions of the modules in question are strictly smaller, so by induction we can assume that there is a trivial complementary submodule Q/S_0 of dimension one so that $R/S_0 = R_0/S_0 \oplus Q/S_0$ (here $Q \subset R$ is a submodule containing S_0, and $\dim Q = \dim S_0 + 1$). Now, since S_0 is irreducible, we can use the weak form of the criterion to write $Q = S_0 \oplus P$ with P trivial one-dimensional submodule of Q. One concludes that $R = R_0 \oplus P$.*

In the proof of Proposition 20, we will consider the A-module of linear maps

$$\mathrm{Hom}(V, W) \qquad : \qquad (a.f)(v) \;=\; \sum_{(a)} a_{(1)}.f(\gamma(a_{(2)}).v) \quad \text{for } a \in A,\; v \in V,\; f \in \mathrm{Hom}(V, W)$$

associated to two A-modules V and W. The subspace $\mathrm{Hom}_A(V, W) \subset \mathrm{Hom}(V, W)$ of A-module maps from V to W is

$$\mathrm{Hom}_A(V, W) \;=\; \{f : V \to W \text{ linear} \mid f(a.v) = a.f(v) \;\text{ for all } v \in V,\, a \in A\}\,.$$

Generally, for any A-module V, the trivial part V^A of V is defined as

$$V^A \;=\; \{v \in V \mid a.v = \epsilon(a)\, v \;\text{ for all }\; a \in A\}\,.$$

The trivial part of the A-module $\mathrm{Hom}(V, W)$ happens to consist precisely of the A-module maps.

Lemma 21. *A map $f \in \mathrm{Hom}(V, W)$ is an A-module map if and only if $a.f = \epsilon(a)\, f$ for all $a \in A$. In other words, we have $\mathrm{Hom}_A(V, W) = \mathrm{Hom}(V, W)^A$.*

Proof. Assuming that f is an A-module map we calculate

$$(a.f)(v) \;=\; \sum_{(a)} a_{(1)}.f(\gamma(a_{(2)}.v)) \;=\; \sum_{(a)} a_{(1)}\gamma(a_{(2)}).f(v) \;=\; \epsilon(a)\, f(v),$$

which shows the "only if" part. To prove the "if" part, suppose that $a.f = \epsilon(a)\, f$ for all $a \in A$. Then calculate

$$\begin{aligned} f(a.v) \;&=\; f\Big(\sum_{(a)} \epsilon(a_{(1)})a_{(2)}.v\Big) \;=\; \sum_{(a)} \epsilon(a_{(1)})\, f(a_{(2)}.v) \\ &=\; \sum_{(a)} (a_{(1)}.f)(a_{(2)}.v) \;=\; \sum_{(a)} a_{(1)}.f\Big(\gamma(a_{(2)})a_{(3)}.v\Big) \\ &=\; \sum_{(a)} a_{(1)}.f(\epsilon(a_{(2)})\, v) \;=\; a.f(v). \end{aligned}$$

□

The observation that allows us to reduce general semisimplicity to the codimension one criterion concerns the module $\mathrm{Hom}(V, W)$ in the particular case when W is a submodule of V. We are searching for an A-linear projection to W.

Lemma 22. *Let V be an A-module and $W \subset V$ a submodule. Let*

$$r : \mathrm{Hom}(V, W) \to \mathrm{Hom}(W, W)$$

be the restriction map given by $r(f) = f|_W$ for all $f : V \to W$. Denote by R the subspace of maps whose restriction is a multiple of the identity of W, that is

$$R = r^{-1}(\mathbb{C}\, \mathrm{id}_W).$$

Then we have

(a) $\mathrm{Im}\,(r|_R) = \mathbb{C}\, \mathrm{id}_W$

(b) $R \subset \mathrm{Hom}(V, W)$ *is a submodule*

(c) $\mathrm{Ker}\,(r|_R) \subset R$ *is a submodule*

(d) $R/\mathrm{Ker}\,(r|_R)$ *is a trivial one dimensional module.*

Proof. The assertion (a) is obvious, since $\mathrm{Im}\,(r|_R) \subset \mathbb{C}\, \mathrm{id}_W$ by definition and the image of any projection $p : V \to W$ is id_W. It follows directly also that $R/\mathrm{Ker}\,(r|_R)$ is a one-dimensional vector space. All the rest of the properties are consequences of the following calculation: if $f \in R$ so that there is a $\lambda \in \mathbb{C}$ such that $f(w) = \lambda\, w$ for all $w \in W$, then for any $a \in A$ we have

$$(a.f)(w) = \sum_{(a)} a_{(1)}.f\Big(\gamma(a_{(2)}).w\Big) = \sum_{(a)} a_{(1)}.\Big(\lambda\, \gamma(a_{(2)}).w\Big) = \lambda \sum_{(a)} a_{(1)}\gamma(a_{(2)}).w = \lambda\epsilon(a)\, w.$$

Indeed, this directly implies (b): $(a.f)|_W = \lambda\epsilon(a)\, \mathrm{id}_W$. For (c), note that $\mathrm{Ker}\,(r)$ corresponds to the case $\lambda = 0$, in which case also $(a.f)|_W = 0$. For (d), rewrite the rightmost expression once more to get $(a.f)|_W = \epsilon(a)\, f|_W$ and thus $a.f = \epsilon(a)\, f + g$ where $g = a.f - \epsilon(a)\, f$ and note that $g|_W = 0$. □

We are now ready to give a proof of the semisimplicity criterion.

Proof of Proposition 20. Assume the property that all codimension one submodules with trivial quotient modules have complements. We will establish semisimplicity by verifying property (iv) of Proposition 19. Suppose therefore that V is a finite dimensional A-module and $W \subset V$ is a submodule. Consider $R \subset \mathrm{Hom}(V, W)$ consisting of those $f : V \to W$ for which the restriction $f|_W$ to W is a multiple of identity, and R_0 consisting of those $f : V \to W$ which are zero on W. By the above lemma $R_0 \subset R \subset \mathrm{Hom}(V, W)$ are submodules and R/R_0 is the one dimensional trivial A-module. By the assumption, then, R_0 has a complementary submodule P, which is one dimensional and trivial. Choose a non-zero $\pi \in P$ normalized so that $\pi|_W = 1\, \mathrm{id}_W$. Then $\pi : V \to W$ is a projection to W. Since P is a trivial module, we have $a.\pi = \epsilon(a)\, \pi$, so by Lemma 21 the projection $\pi : V \to W$ is an A-module map. Thus property (iv) of Proposition 19 holds, and by the same Proposition, A is semisimple. □

Let us then check that $\mathcal{U}_q(\mathfrak{sl}_2)$ satisfies the criterion when q is not a root of unity.

Lemma 23. *Let $q \in \mathbb{C} \setminus \{0\}$ not a root of unity. If V is a finite dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module and $W \subset V$ is an irreducible submodule such that V/W is a trivial one dimensional module, then there is a trivial one dimensional submodule $W' \subset V$ such that $V = W \oplus W'$.*

Proof. Theorem 18 lists all possible irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-modules, they are W_d^ε for d a positive integer and $\varepsilon \in \{\pm 1\}$. So we have $W \cong W_d^\varepsilon$ for some d and ε. We first suppose that $d \neq 1$ or $\varepsilon \neq +1$ — the case when W also is trivial (i.e. $W \cong W_1^{+1}$) is treated separately. By Equation (4.23), the central element C acts as multiplication by the constant $c_{d,\varepsilon} = \varepsilon(q^d + q^{-d})/(q - q^{-1})^2$ on W. On the quotient V/W it acts as $c_{1,1} = (q + q^{-1})/(q - q^{-1})^2$. Therefore

$$\frac{1}{c_{d,\varepsilon} - c_{1,1}}\Big(C - c_{1,1}\,\mathrm{id}_V\Big)$$

is a projection to W which is also an $\mathcal{U}_q(\mathfrak{sl}_2)$-module map. This implies that W has a complementary submodule $\mathrm{Ker}\,(C - c_{1,1}\,\mathrm{id})$.

The case when both W and V/W are trivial has to be treated separately, but it is very easy to show that in this case V is a trivial 2-dimensional representation and any complementary subspace to W is a complementary submodule. □

Corollary 24. *For q not a root of unity, the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is semisimple.*

Proof. Use Proposition 20, Remark 3 and Lemma 23. □

Solutions to YBE from infinite dimensional Drinfeld doubles

Let us pause for a moment to see where we are in finding solutions to the Yang-Baxter equation, Equation (4.10). The overall story goes smoothly — by Theorem 6 any representation of any braided Hopf algebra gives us a solution of YBE, and by Theorem 10 the Drinfeld double construction produces braided Hopf algebras. We have even concretely described an interesting Drinfeld double D_{q^2} and a quotient $\mathcal{U}_q(\mathfrak{sl}_2)$ of it, and we have found all their irreducible representations in Theorems 17 and 18.

There is just one issue — to obtain the universal R-matrix which makes the Drinfeld double a braided Hopf algebra, we had to assume finite dimensionality of the Hopf algebra whose Drinfeld double we take. Unfortunately, the Hopf algebra D_{q^2} is a Drinfeld double of the infinite dimensional building block Hopf algebra H_{q^2}, so we seem to have a small problem.

Although we can't properly make D_{q^2} and $\mathcal{U}_q(\mathfrak{sl}_2)$ braided Hopf algebras, in that we will not really find a universal R-matrix in the second tensor power of these algebras, we can nevertheless find solutions of the Yang-Baxter equation by more or less the same old receipe. Let us first describe the heuristics, and then prove the main result, and finally give example applications with the representations of D_{q^2} and $\mathcal{U}_q(\mathfrak{sl}_2)$.

Heuristics and formula for the R-matrices

Assume that A is a Hopf algebra with invertible antipode, and $\mathcal{D} = \mathcal{D}(A, A^\circ)$ is the Drinfeld double. Recall that $\mathcal{D} = A \otimes A^\circ$ as a vector space, and the Hopf algebras A and $(A^\circ)^{\mathrm{cop}}$ are embedded to $\mathcal{D}$ by the maps

$$\begin{aligned} \iota_A &: A \to \mathcal{D} & \qquad \iota_{A^\circ} &: A^\circ \to \mathcal{D} \\ & a \mapsto a \otimes 1^* & & \varphi \mapsto 1 \otimes \varphi. \end{aligned}$$

We would like to set, as in Theorem 10,

$$R \overset{?}{=} \sum_\alpha \iota_A(e_\alpha) \otimes \iota_{A^\circ}(\delta^\alpha), \tag{4.26}$$

where (e_α) is a basis of A, and (δ^α) is a "dual basis" of A°. This is of course problematic in the infinite dimensional case.

Let us first fix some notation. Since A embeds to $\mathcal{D}$ as a Hopf algebra, we can consider restrictions on A of elements $\phi \in \mathcal{D}^\circ$ of the restricted dual of the Drinfeld double: define $\phi|_A \in A^\circ$ by

$$\langle \phi|_A, a \rangle \; = \; \langle \phi, \iota_A(a) \rangle \qquad \text{for all } a \in A.$$

Furthermore, since A° embeds to $\mathcal{D}$, we can interpret the above as an element of $\mathcal{D}$. We define

$$\phi' \; = \; \iota_{A^\circ}(\phi|_A) \in \mathcal{D} \qquad \text{for any } \phi \in \mathcal{D}^\circ. \tag{4.27}$$

If the bases (e_α) and (δ^α) were to be dual to each other, we would expect a formula of the type

$$\sum_\alpha \langle \varphi, e_\alpha \rangle \, \delta^\alpha \; \overset{?}{=} \; \varphi$$

to hold for any $\varphi \in A^\circ$. So in particular when $\varphi = \phi|_A$, we expect

$$\sum_\alpha \langle \phi|_A, e_\alpha \rangle \, \iota_B(\delta^\alpha) \; \overset{?}{=} \; \iota_B(\phi|_A) \; = \; \phi'.$$

Returning to the heuristic formula (4.26) for the universal R-matrix of $\mathcal{D}$, let us consider how it would act on representations. If V is a $\mathcal{D}$-module with basis $(v_j)_{j=1}^d$ and representative forms $\lambda_{i,j} \in \mathcal{D}^\circ$ such that

$$x.v_j \; = \; \sum_{i=1}^d \langle \lambda_{i,j}, x \rangle \, v_i \qquad \text{for any } x \in \mathcal{D}$$

we would like to make the R-matrix act on $V \otimes V$ by

$$\begin{aligned} R(v_i \otimes v_j) \; &\overset{?}{=} \; \sum_\alpha \iota_A(e_\alpha).v_i \otimes \iota_{A^\circ}(\delta^\alpha).v_j \\ &\overset{?}{=} \; \sum_\alpha \sum_{l,k=1}^d \underbrace{\langle \lambda_{l,i}, \iota_A(e_\alpha) \rangle}_{=\langle \lambda_{l,i}|_A, e_\alpha \rangle} \langle \lambda_{k,j}, \iota_{A^\circ}(\delta^\alpha) \rangle \, v_l \otimes v_k \\ &\overset{?}{=} \; \sum_{l,k=1}^d \langle \lambda_{k,j}, (\lambda_{l,i})' \rangle \, v_l \otimes v_k. \end{aligned}$$

We have found a formula that is expressed only in terms of the representative forms, and therefore it is meaningful also when A is infinite dimensional. We are mostly using $\check{R} = S_{V,V} \circ R$, so the appropriate definitions are

$$\check{R} : V \otimes V \to V \otimes V \qquad \check{R}(v_i \otimes v_j) \; = \; \sum_{k,l=1}^d r_{i,j}^{k,l} \, v_k \otimes v_l$$
$$r_{i,j}^{k,l} \; = \langle \lambda_{k,j}, (\lambda_{l,i})' \rangle. \tag{4.28}$$

Proving that the formula gives solutions to YBE

We now check that Equation (4.28) indeed works. We record a small lemma, which is needed along the way.

Lemma 25. *For any $\phi \in \mathcal{D}^\circ$ and $x \in \mathcal{D}$, the following equality holds in $\mathcal{D}$*

$$\sum_{(\phi)} \sum_{(x)} \langle \phi_{(1)}, x_{(2)} \rangle \, x_{(1)} (\phi_{(2)})' \; = \; \sum_{(\phi)} \sum_{(x)} \langle \phi_{(2)}, x_{(1)} \rangle \, (\phi_{(1)})' x_{(2)}.$$

When $x = \psi'$ with $\psi \in \mathcal{D}^\circ$ we have

$$\sum_{(\phi),(psi)} \langle \phi_{(1)}, (\psi_{(1)})' \rangle \, (\psi_{(2)})' \, (\phi_{(2)})' \; = \; \sum_{(\phi),(\psi)} \langle \phi_{(2)}, (\psi_{(2)})' \rangle \, (\phi_{(1)})' (\psi_{(1)})'.$$

Proof. The proof of the first statement is an exercise. The second statement follows as a particular case of the first, when we observe that for $x = \psi'$ the coproduct of x can be written in terms of the coproduct of ψ as

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} = \Delta_{\mathcal{D}}(x) = \Delta_{\mathcal{D}}(\iota_{A^\circ}(\psi|_A)) = (\iota_{A^\circ} \otimes \iota_{A^\circ})((\mu^*)^{\mathrm{cop}}(\psi|_A)) = \sum_{(\psi)} (\psi_{(2)})' \otimes (\psi_{(1)})'.$$

□

Theorem 26. *Let A be a Hopf algebra with invertible antipode and $B \subset A^\circ$ a Hopf subalgebra of the restricted dual, and let $\mathcal{D} = \mathcal{D}(A, B)$ be the Drinfeld double associated to A and B. Let V be a $\mathcal{D}$-module with basis $(v_j)_{j=1}^d$, and assume that the representative forms $\lambda_{i,j} \in \mathcal{D}^\circ$ satisfy $\lambda_{i,j}|_A \in B$. Then the linear map $\check{R} : V \otimes V \to V \otimes V$ defined by Equation (4.28) satisfies the Yang-Baxter equation (4.10). Furthermore, the associated braid group representation on $V^{\otimes n}$ commutes with the action of $\mathcal{D}$.*

Proof. The proof is a direct calculation — besides the definitions, the key properties to keep in mind are the coproduct formula of representative forms $\mu^*(\lambda_{i,j}) = \sum_k \lambda_{i,k} \otimes \lambda_{k,j}$ and the formulas of Lemma 25. Let us take an elementary tensor $v_s \otimes v_t \otimes v_u \in V \otimes V \otimes V$. Applying the left hand side of the YBE on this, we get

$$\begin{aligned}
& \check{R}_{12} \circ \check{R}_{23} \circ \check{R}_{12}(v_s \otimes v_t \otimes v_u) \\
= & \sum_{i,j,k,l,m,n} r^{l,m}_{i,k}\, r^{k,n}_{j,u}\, r^{i,j}_{s,t}\; v_l \otimes v_m \otimes v_n \\
= & \sum_{i,j,k,l,m,n} \langle \lambda_{l,k}, (\lambda_{m,i})' \rangle\, \langle \lambda_{k,u}, (\lambda_{n,j})' \rangle\, \langle \lambda_{i,t}, (\lambda_{j,s})' \rangle\; v_l \otimes v_m \otimes v_n \\
= & \sum_{i,j,l,m,n} \langle \lambda_{l,u}, (\lambda_{m,i})'(\lambda_{n,j})' \rangle\, \langle \lambda_{i,t}, (\lambda_{j,s})' \rangle\; v_l \otimes v_m \otimes v_n \\
= & \sum_{l,m,n} \sum_{(\lambda_{m,t}),(\lambda_{n,s})} \langle \lambda_{l,u}, ((\lambda_{m,t})_{(1)})'((\lambda_{n,s})_{(1)})' \rangle\, \langle (\lambda_{m,t})_{(2)}, ((\lambda_{n,s})_{(2)})' \rangle\; v_l \otimes v_m \otimes v_n,
\end{aligned}$$

where in the last two steps we used the coproduct formula for representative forms. Similarly, the right hand side of the YBE takes the value

$$\begin{aligned}
& \check{R}_{23} \circ \check{R}_{12} \circ \check{R}_{23}(v_s \otimes v_t \otimes v_u) \\
= & \sum_{i,j,k,l,m,n} r^{m,n}_{k,j}\, r^{l,k}_{s,i}\, r^{i,j}_{t,u}\; v_l \otimes v_m \otimes v_n \\
= & \sum_{i,j,k,l,m,n} \langle \lambda_{m,j}, (\lambda_{n,k})' \rangle\, \langle \lambda_{l,i}, (\lambda_{k,s})' \rangle\, \langle \lambda_{i,u}, (\lambda_{j,t})' \rangle\; v_l \otimes v_m \otimes v_n \\
= & \sum_{j,k,l,m,n} \langle \lambda_{m,j}, (\lambda_{n,k})' \rangle\, \langle \lambda_{l,u}, (\lambda_{k,s})'(\lambda_{j,t})' \rangle\; v_l \otimes v_m \otimes v_n \\
= & \sum_{l,m,n} \sum_{(\lambda_{m,t}),(\lambda_{n,s})} \langle (\lambda_{m,t})_{(1)}, ((\lambda_{n,s})_{(1)})' \rangle\, \langle \lambda_{l,u}, ((\lambda_{n,s})_{(2)})'((\lambda_{m,t})_{(2)})' \rangle\; v_l \otimes v_m \otimes v_n.
\end{aligned}$$

The equality of the two sides of the Yang-Baxter equation then follows from the second formula of Lemma 25 above.

To prove that the associated braid group representation commutes with the action of $\mathcal{D}$, it is enough to show that on $V \otimes V$ the matrix $\check{R}$ commutes with the action of $\mathcal{D}$. Let $x \in \mathcal{D}$, and calculate on elementary tensors

$$
\begin{aligned}
x.\Big(\check{R}(v_i \otimes v_j)\Big) &= x.\Big(\sum_{k,l} \langle \lambda_{k,j}, (\lambda_{l,i})' \rangle \; v_k \otimes v_l\Big) \\
&= \sum_{(x)} \sum_{k,l} \langle \lambda_{k,j}, (\lambda_{l,i})' \rangle \; \Big(x_{(1)}.v_k \otimes x_{(2)}.v_l\Big) \\
&= \sum_{(x)} \sum_{k,l,m,n} \langle \lambda_{k,j}, (\lambda_{l,i})' \rangle \; \langle \lambda_{m,k}, x_{(1)} \rangle \; \langle \lambda_{n,l}, x_{(2)} \rangle \; v_m \otimes v_n \\
&= \sum_{(x)} \sum_{l,m,n} \langle \lambda_{m,j}, x_{(1)} \, (\lambda_{l,i})' \rangle \; \langle \lambda_{n,l}, x_{(2)} \rangle \; v_m \otimes v_n \\
&= \sum_{(x)} \sum_{(\lambda_{n,i})} \sum_{m,n} \langle \lambda_{m,j}, x_{(1)} \, ((\lambda_{n,i})_{(2)})' \rangle \; \langle (\lambda_{n,i})_{(1)}, x_{(2)} \rangle \; v_m \otimes v_n.
\end{aligned}
$$

This is to be compared with

$$
\begin{aligned}
\check{R}\Big(x.(v_i \otimes v_j)\Big) &= \sum_{(x)} \check{R}\Big(x_{(1)}.v_i \otimes x_{(2)}.v_j\Big) \\
&= \sum_{(x)} \sum_{k,l} \langle \lambda_{k,i}, x_{(1)} \rangle \; \langle \lambda_{l,j}, x_{(2)} \rangle \; \check{R}(v_k \otimes v_l) \\
&= \sum_{(x)} \sum_{k,l,m,n} \langle \lambda_{k,i}, x_{(1)} \rangle \; \langle \lambda_{l,j}, x_{(2)} \rangle \; \langle \lambda_{m,l}, (\lambda_{n,k})' \rangle \; v_m \otimes v_n \\
&= \sum_{(x)} \sum_{k,m,n} \langle \lambda_{k,i}, x_{(1)} \rangle \; \langle \lambda_{m,j}, (\lambda_{n,k})' \, x_{(2)} \rangle \; v_m \otimes v_n \\
&= \sum_{(x)} \sum_{(\lambda_{n,i})} \sum_{m,n} \langle (\lambda_{n,i})_{(2)}, x_{(1)} \rangle \; \langle \lambda_{m,j}, ((\lambda_{n,i})_{(1)})' \, x_{(2)} \rangle \; v_m \otimes v_n.
\end{aligned}
$$

The two expressions agree by virtue of Lemma 25. □