$$\boxed{\tilde\beta\beta = \beta\tilde\beta + A_0}$$

$\Downarrow$

$$\tilde\beta^2\beta = \tilde\beta(\beta\tilde\beta + A_0)$$
$$= (\beta\tilde\beta + A_0)\tilde\beta + A_2\tilde\beta$$
$$= \beta\tilde\beta^2 + (A_0 + A_2)\tilde\beta$$
$$= \beta\tilde\beta^2 + [2]_q \cdot A_1\tilde\beta$$

<u>Def</u>: $A_n := q^n\alpha - q^{-n}\tilde\alpha$

<u>Note</u>: $\tilde\beta A_n = A_{n+2}\tilde\beta$

$A_n\beta = \beta A_{n+2}$

Consider the following elements in the ideal generated by $\tilde\beta^2$:

$$\beta^2\tilde\beta^2$$

$$\beta\tilde\beta^2\beta = \beta^2\tilde\beta^2 + [2]_q\,\beta A_1\tilde\beta$$

$$\tilde\beta^2\beta^2 = (\beta\tilde\beta^2 + [2]_q\,A_1\tilde\beta)\beta$$
$$= [2]_q\,A_1(\beta\tilde\beta + A_0) + \beta(\beta\tilde\beta^2 + [2]_q A_1\tilde\beta)$$
$$= \beta^2\tilde\beta^2 + [2]_q\,\beta(A_3 + A_1)\tilde\beta + [2]_q\,A_1A_0$$
$$= \beta^2\tilde\beta^2 + [2]_q^2\cdot\beta A_2\tilde\beta + [2]_q\,A_1A_0$$

Now we multiply from left by well chosen polynomials

$$A_{-1}\tilde\beta^2\beta^2 = \beta^2A_3\tilde\beta^2 + [2]_q^2\cdot\beta A_2A_1\tilde\beta + [2]_q\underbrace{A_1A_0A_{-1}}_{\text{this is the polynom. we want in the ideal}}$$

$$A_0\beta\tilde\beta^2\beta = \beta^2A_4\tilde\beta^2 + [2]_q\cdot\beta\underbrace{A_2A_1}\tilde\beta$$

this can be cancelled with a similar term in $A_{-1}\tilde\beta^2\beta^2$

So: $$\frac{1}{[2]_q}A_{-1}\tilde\beta^2\beta^2 - A_0\beta\tilde\beta^2\beta$$
$$= \beta^2\left(\frac{1}{[2]_q}A_3 - A_4\right)\tilde\beta^2 + A_1A_0A_{-1}$$

And we get:

$$\frac{1}{[2]_q}A_{-1}\tilde\beta^2\beta^2 - A_0\beta\tilde\beta^2\beta + \left(A_0 - \frac{1}{[2]_q}A_{-1}\right)\beta^2\tilde\beta^2 = A_1A_0A_{-1}$$