Hopf algebras and representations

Spring 2011

Problem sheet 7

Model solutions

Ex 1

Let's consider an algebra with generators a, b and relation $ab = q\,ba$ where $q^p = 1$ and $q^j \neq 1$, $j = 1, 2, \ldots, p-1$.
Note that $a^p b = b a^p$, $a b^p = b^p a$ and hence a^p, b^p are in the center of the algebra.
Write any $n \in \mathbb{N}$ as $n = p\,D(n) + R(n)$, $D(n) \in \mathbb{N}$, $R(n) \in \{0, 1, \ldots, p-1\}$. Now

$$(a+b)^n = (a+b)^{p D(n)} (a+b)^{R(n)} = (a^p + b^p)^{D(n)} (a+b)^{R(n)}$$

(↑ Problem sheet 4 Exercise 1)

$$= \left[\sum_{i=0}^{D(n)} \binom{D(n)}{i} b^{pi} a^{pD(n)-pi} \right] \cdot \left[\sum_{j=0}^{R(n)} \left[\!\left[\begin{matrix} R(n) \\ j \end{matrix} \right]\!\right]_q b^j a^{R(n)-j} \right]$$

$$= \sum_{k=0}^{n} \mathbb{1}_{R(k) \le R(n)} \cdot \binom{D(n)}{D(k)} \left[\!\left[\begin{matrix} R(n) \\ R(k) \end{matrix} \right]\!\right]_q \cdot b^k a^{n-k}$$

which you can compare with Problem sheet 4, Exercise 1 to get the claimed formula.

Ex 3

(a) Let $a, a' \in A$ and let $\varphi \in B^{\circ}$. Write $\mu_B^*(\varphi) = \sum_{i=1}^{l} g_i \otimes h_i$.

$$\langle \mu_A^*(f^*(\varphi)), a \otimes a' \rangle = \langle f^*(\varphi), aa' \rangle$$

$$= \langle \varphi, f(a) f(a') \rangle = \langle \mu_B^*(\varphi), f(a) \otimes f(b) \rangle$$

$$= \sum_{i=1}^{l} \langle g_i, f(a) \rangle \langle h_i, f(a') \rangle$$

$$= \left\langle \sum_{i=1}^{l} f^*(g_i) \otimes f^*(h_i), a \otimes a' \right\rangle$$

(b) Since $\varphi \in B^{\circ}$, $g_i \in B^{\circ}$, $h_i \in B^{\circ}$, we can restrict above $f^*|_{B^{\circ}}$. Since $f^*(\varphi) \in A^{\circ}$, we can restrict $\mu_A^*|_{A^{\circ}}$. Therefore the above can be written as

$$\Delta_{A^{\circ}} \circ f^{\circ} = (f^{\circ} \otimes f^{\circ}) \circ \Delta_{B^{\circ}}$$

where $\Delta_{A^{\circ}} = \mu_A^*|_{A^{\circ}}$, $\Delta_{B^{\circ}} = \mu_B^*|_{B^{\circ}}$ and $f^{\circ} = f^*|_{B^{\circ}}$.

Also

$$\eta_A^* \circ f^*(\varphi) = \langle f^*(\varphi), 1_A \rangle = \langle \varphi, 1_B \rangle = \eta_B^*(\varphi)$$

and we can restrict $\varepsilon_{A^{\circ}} = \eta_A^*|_{A^{\circ}}$, $\varepsilon_{B^{\circ}} = \eta_B^*|_{B^{\circ}}$.

Exercise 2

In the group algebra $A = \mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[t, t^{-1}]$ equipped with the Hopf algebra structure such that $\Delta(t) = t \otimes t$, we define the elements g_z ($z \in \mathbb{C}^*$) and s of the restricted dual A° by

$$\langle g_z, t^m \rangle = z^m \tag{0.1}$$
$$\langle s, t^m \rangle = m \tag{0.2}$$

for all $m \in \mathbb{Z}$.

Let V be a vector space with a basis $(e_i)_{i=1}^n$ and an A-module structure given by

$$t \cdot e_i = z e_i + e_{i-1}$$

where $z \in \mathbb{C}^*$ and e_0 is interpreted as $0 \in V$. The representative forms $\lambda_{i,j} \in A^\circ$ are defined so that

$$a \cdot e_j = \sum_{i=1}^{n} \langle \lambda_{i,j}, a \rangle \, e_i$$

for all $a \in A$.

Claim 1. *We have*

$$\lambda_{i,j} = \begin{cases} 0 & \text{if } i > j \\ g_z & \text{if } i = j \\ \frac{z^{i-j}}{(j-i)!}\, s(s-1)\cdots(s+i-j+1) g_z & \text{if } i < j \end{cases} .$$

Starting from the definition

$$t \cdot e_j = z e_j + e_{j-1},$$

one finds recursively the action of t^m, $m \in \mathbb{N}$, on the basis vectors

$$\begin{aligned} t^2 \cdot e_j &= t \cdot \big(z e_j + e_{j-1}\big) = z^2 e_j + 2z e_{j-1} + e_{j-2} \\ t^3 \cdot e_j &= t \cdot \big(z^2 e_j + 2z e_{j-1} + e_{j-2}\big) = z^3 e_j + 3z^2 e_{j-1} + 3z e_{j-2} + e_{j-3} \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

et cetera, where $e_i = 0$ whenever $i \leq 0$. By a simple induction one proves that for $m \in \mathbb{N}$

$$t^m \cdot e_j = \sum_{r=0}^{m} \binom{m}{r} z^{m-r}\, e_{j-r} = \sum_{i=1}^{j} \binom{m}{j-i} z^{m+i-j}\, e_i.$$

Therefore we have for $m \in \mathbb{N}$

$$\langle \lambda_{i,j}, t^m \rangle = \binom{m}{j-i} z^{m+i-j} = \frac{1}{(j-i)!} m(m-1)\cdots(m+i-j+1)\, z^{m+i-j}.$$

Recalling that $(\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \cdots \circ (\Delta \otimes \mathrm{id}) \circ \Delta(t^m) = t^m \otimes \cdots \otimes t^m$ and $\langle 1_{A^\circ}, t^m \rangle = \epsilon(t^m) = 1$, we should compare the above expression with

$$\begin{aligned} \langle s(s-1)\cdots(s+i-j+1) g_z, t^m \rangle &= \langle s \otimes (s-1) \otimes \cdots \otimes (s+i-j+1) \otimes g_z \,,\; t^m \otimes \cdots \otimes t^m \rangle \\ &= \langle s, t^m \rangle \langle s-1, t^m \rangle \cdots \langle s+i-j+1, t^m \rangle \langle g_z, t^m \rangle \\ &= m(m-1)\cdots(m+i-j+1)\, z^m. \end{aligned}$$

We indeed observe the claimed formula for $m \in \mathbb{N}$

$$\langle \lambda_{i,j}, t^m \rangle = \frac{z^{i-j}}{(j-i)!} \langle s(s-1)\cdots(s+i-j+1) g_z, t^m \rangle.$$

We should still verify the formula for $m < 0$. To this end we first need the matrix of t^{-1} acting on V

$$t^{-1} \cdot e_j \;=\; \sum_{i=1}^{n} \left\langle \lambda_{i,j}, t^{-1} \right\rangle e_i .$$

By the A-module property it has to be the inverse of the matrix of t

$$\begin{aligned} e_j \;&= tt^{-1} \cdot e_j \;=\; \sum_{i=1}^{n} \left\langle \lambda_{i,j}, t^{-1} \right\rangle \left(z e_i + e_{i-1} \right) \\ &= \sum_{i'=1}^{n} \Big(\underbrace{z \left\langle \lambda_{i',j}, t^{-1} \right\rangle + \left\langle \lambda_{i'+1,j}, t^{-1} \right\rangle}_{= \delta_{i',j}} \Big) e_{i'} \end{aligned}$$

so with a small recursive calculation we find the formula

$$\left\langle \lambda_{i,j}, t^{-1} \right\rangle \;=\; \begin{cases} 0 & \text{if } i > j \\ z^{-1} & \text{if } i = j \\ (-1)^{j-i} z^{i-j-1} & \text{if } i < j \end{cases}$$

It is convenient for calculations to write this as

$$t^{-1} \cdot e_j \;=\; \sum_{r \geq 0} \frac{(-1)^r}{z^{1+r}} \, e_{j-r} .$$

Now it is a combinatorial exercise to find the action of t^{-m} for $m \in \mathbb{N}$, namely

$$\begin{aligned} t^{-m} \cdot e_j \;&=\; \sum_{r_1 \geq 0} \cdots \sum_{r_m \geq 0} \frac{(-1)^{r_1 + \cdots + r_m}}{z^{m + r_1 + \cdots + r_m}} \, e_{j - r_1 - \cdots - r_m} \\ &= \sum_{r \geq 0} \frac{(-1)^r}{z^{m+r}} \binom{r+m-1}{m-1} e_{j-r}, \end{aligned}$$

where we used the observation that the number of m-tuples $(r_1, \ldots, r_m)$ of non-negative integers such that $r_1 + \cdots + r_m = r$ is the same as the number of ways of placing $m-1$ separators between r boxes, or the number of ways of choosing among $r+m-1$ boxes $m-1$ which serve as separators.[1] The non-zero matrix elements of t^{-m} are thus when $i \leq j$

$$\begin{aligned} \left\langle \lambda_{i,j}, t^{-m} \right\rangle \;&=\; \frac{(-1)^{j-i}}{z^{m+j-i}} \binom{m+j-i-1}{m-1} \;=\; \frac{(-1)^{j-i}}{z^{m+j-i}} \frac{1}{(j-i)!} m(m+1) \cdots (m+j-i-1) \\ &= \frac{z^{i-j}}{(j-i)!} (-m)(-m-1) \cdots (-m-j+i+1) \, z^{-m} \end{aligned}$$

which again coincides with

$$\frac{z^{i-j}}{(j-i)!} \big\langle s(s-1) \cdots (s+i-j+1) g_z, t^{-m} \big\rangle .$$

By considering the values of the representative forms on the basis $(t^m)_{m \in \mathbb{Z}}$ of A we have therefore shown

$$\begin{aligned} \lambda_{i,j} \;&=\; \frac{z^{i-j}}{(j-i)!} s(s-1) \cdots (s+i-j+1) g_z \qquad && \text{when } i \leq j \text{ and} \\ \lambda_{i,j} \;&=\; 0 && \text{when } i > j. \end{aligned}$$

[1] It is a pleasant exercise also to derive the combinatorial identity using generating functions. Setting $G_m(q) = \left(\sum_{s \geq 0} q^s \right)^m = (1-q)^{-m}$, the quantity we're interested in is the coefficient of q^r in the power series expansion of $G_m(q)$ at $q = 0$. But $(1-q)^{-m}$ is proportional to the $(m-1)^{\text{th}}$ derivative of the geometric series $(1-q)^{-1}$, so the coefficient is easy to figure out.

Exercise 4

Definition 1. *Given two Hopf algebras $(A_i, \mu_i, \Delta_i, \eta_i, \epsilon_i, \gamma_i)$, $i = 1, 2$, we can equip $A_1 \otimes A_2$ with a Hopf algebra structure given by*

$$\begin{aligned} \mu &= (\mu_1 \otimes \mu_2) \circ (\mathrm{id}_{A_1} \otimes S_{A_2,A_1} \otimes \mathrm{id}_{A_2}) \\ \Delta &= (\mathrm{id}_{A_1} \otimes S_{A_1,A_2} \otimes \mathrm{id}_{A_2}) \circ (\Delta_1 \otimes \Delta_2) \\ \eta &= \eta_1 \otimes \eta_2 \\ \epsilon &= \epsilon_1 \otimes \epsilon_2 \\ \gamma &= \gamma_1 \otimes \gamma_2. \end{aligned}$$

Claim 2. *Let $A = \mathbb{C}[x]$ be the algebra of polynomials equipped with the unique Hopf algebra structure such that $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then there is an isomorphism of Hopf algebras*

$$A^\circ \;\cong\; A \otimes \mathbb{C}[\mathbb{C}].$$

We will describe explicitly both A° and $A \otimes \mathbb{C}[\mathbb{C}]$. First, however, let us see what is the Hopf algebra structure of A.

Hopf algebra structure of $A = \mathbb{C}[x]$

Note that while the algebra $A = \mathbb{C}[x]$ of polynomials in resembles the algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials considered in the previous exercises, the Hopf algebra structure turns out to be very different.

Multiplication and unit are those of the polynomial algebra, and in terms of the linear basis $(x^n)_{n\in\mathbb{N}}$ of $\mathbb{C}[x]$ they read $\mu(x^n \otimes x^m) = x^{n+m}$ and $\eta(\lambda) = \lambda x^0$. The coproduct must be an algebra morphism, it is determined by its value $\Delta(x) = x \otimes 1 + 1 \otimes x$ at the generator (and since A is the free algebra generated by x, such an algebra morphism exists). By induction one finds the formula

$$\Delta(x^n) = \sum_{j=0}^{n} \binom{n}{j} \, x^j \otimes x^{n-j}.$$

Since x is a primitive element, we must have $\epsilon(x) = 0$, and again the counit should be an algebra morphism, it is determined by this value at the generator and we get

$$\epsilon(x^n) = \delta_{n,0}.$$

The fact that x is primitive implies also $\gamma(x) = -x$, and since γ should be an algebra morphism $A \to A^{\mathrm{op}}$ it, too, is fixed by the value at the generator. In general we have

$$\gamma(x^n) = (-1)^n x^n.$$

Note that axioms (1) and (2) of Hopf algebras express the fact that (A, μ, η) is an algebra, and they are therefore clearly satisfied in the present case. Axioms (4), (5), (5'), (6) express the facts that Δ and ϵ are algebra morphisms, and are therefore satisfied by construction. Axiom (1') states the coassociativity $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ as mappings between $A \to A \otimes A \otimes A$, and axiom (2') states the counitality $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$ as mappings between $A \to \mathbb{C}$. Since all mappings involved are algebra morphisms, it suffices to verify properties (1') and (2') for the generator x, which is easily done. Therefore $(A, \mu, \Delta, \eta, \epsilon)$ is a bialgebra. To prove the remaining property (3) we use the following useful observation.

Lemma 1. *Let $(A, \mu, \Delta, \eta, \epsilon)$ be a bialgebra, which as an algebra is generated by $(g_i)_{i\in I}$. Suppose $\gamma : A \to A^{\mathrm{op}}$ is an algebra morphism such that*

$$\mu \circ (\gamma \otimes \mathrm{id}) \circ \Delta(a) = \eta \circ \epsilon(a) = \mu \circ (\mathrm{id} \otimes \gamma) \circ \Delta(a) \tag{0.3}$$

for all generators $a = g_i$ $(i \in I)$. Then $(A, \mu, \Delta, \eta, \epsilon, \gamma)$ is a Hopf algebra.

Proof. We must establish the axiom (3), $\mu\circ(\gamma\otimes\mathrm{id})\circ\Delta = \eta\circ\epsilon = \mu\circ(\mathrm{id}\otimes\gamma)\circ\Delta$. We will prove that the set of elements $a \in A$ satisfying the relation (0.3) is a subalgebra of A. Since by assumption it contains the generators g_i of A, the statement follows.

In view of the facts $\Delta(1) = 1 \otimes 1$, $\epsilon(1) = 1$ and $\gamma(1) = 1$, which are evident given the algebra morphism properties, the unit $1 \in A$ satisfies (0.3). Suppose now that two elements $a, b \in A$ satisfy (0.3). Write their coproducts using the Sweedler's sigma notation, $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$. Then

$$\begin{aligned}
\mu\circ(\gamma\otimes\mathrm{id})\circ\Delta(ab) &= \mu\circ(\gamma\otimes\mathrm{id})\Big(\sum_{(a),(b)} (a_{(1)}b_{(1)})\otimes(a_{(2)}b_{(2)})\Big) \\
&= \mu\Big(\sum_{(a),(b)} (\gamma(b_{(1)})\gamma(a_{(1)}))\otimes(a_{(2)}b_{(2)})\Big) \\
&= \sum_{(a),(b)} \gamma(b_{(1)})\gamma(a_{(1)})a_{(2)}b_{(2)} \\
&= \epsilon(a) \sum_{(b)} \gamma(b_{(1)})b_{(2)} \;=\; \varepsilon(a)\varepsilon(b)\,1 \;=\; \varepsilon(ab)\,1.
\end{aligned}$$

The other equality is proved by an analogous calculation. □

In view of the Lemma, it is sufficient to verify property (3) for the generator x,

$$\mu\circ(\gamma\otimes\mathrm{id})\circ\Delta(x) \;=\; (-x)1 + (1)x \;=\; 0 \;=\; \epsilon(x)1_A \;=\; \cdots \;=\; \mu\circ(\mathrm{id}\otimes\gamma)\circ\Delta(x).$$

So $A = \mathbb{C}[x]$ indeed has a unique Hopf algebra structure with the given coproduct.

Explicit description of $A \otimes \mathbb{C}[\mathbb{C}]$

A linear basis of A is $(x^n)_{n\in\mathbb{N}}$, and a linear basis of the group algebra $\mathbb{C}[\mathbb{C}]$ of the additive group of complex numbers is $(v_z)_{z\in\mathbb{C}}$. The vector space $A \otimes \mathbb{C}[\mathbb{C}]$ has a basis

$$(x^n \otimes v_z)_{(n,z)\in\mathbb{N}\times\mathbb{C}}$$

and we compute the values of $\mu, \Delta, \eta, \epsilon, \gamma$ on these basis elemensts using the definition of tensor product of Hopf algebras and the known Hopf algebra structures of A and $\mathbb{C}[\mathbb{C}]$. We get

(i) $\mu\big((x^n \otimes v_z)\otimes(x^m \otimes v_w)\big) = x^{n+m} \otimes v_{z+w}$

(ii) $\Delta(x^n \otimes v_z) = \sum_{j=0}^{n} \binom{n}{j} (x^j \otimes v_z)\otimes(x^{n-j}\otimes v_z)$

(iii) $\eta(1) = x^0 \otimes v_0$

(iv) $\epsilon(x^n \otimes v_z) = \delta_{n,0}$

(v) $\gamma(x^n \otimes v_z) = (-1)^n(x^n \otimes v_{-z})$.

Explicit description of the restricted dual A°

A linear basis of A° is given by the derivatives of the evaluations $\mathrm{ev}_z^{(n)}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$ defined as

$$\big\langle \mathrm{ev}_z^{(n)}, p\big\rangle \;=\; p^{(n)}(z)$$

for $p \in \mathbb{C}[x]$ a polynomial and $p^{(n)}$ its n^{th} derivative. This can be shown by methods similar to the analysis of $\mathbb{C}[\mathbb{Z}]^\circ$ in the lectures, or by considering certain recursive sequences as was indicated in the exercise session of 1.4.2010.[2]

The restricted dual A° has a Hopf algebra structure and below we compute the values of the structure constants in this basis.

[2] To be explicit, the *poisson d'avril* went as follows. Elements of the dual $f \in A^*$ correspond to sequences $(f_n)_{n\in\mathbb{N}}$ through $f_n = \langle f, x^n\rangle$. Any element of the restricted dual $f \in A^\circ$ corresponds to a sequence that satisfies a recursion

$$f_{n+m} = \sum_{j=0}^{m-1} c_j f_{n+j} \qquad \forall n \in \mathbb{N}$$

(i) The product of A° is the adjoint of Δ, so we compute directly

$$\begin{aligned}
\left\langle \Delta^*\left(\mathrm{ev}_z^{(n)} \otimes \mathrm{ev}_w^{(m)}\right), x^k \right\rangle &= \left\langle \mathrm{ev}_z^{(n)} \otimes \mathrm{ev}_w^{(m)}, \Delta(x^k) \right\rangle \\
&= \left\langle \mathrm{ev}_z^{(n)} \otimes \mathrm{ev}_w^{(m)}, \sum_{j=0}^{k} \binom{k}{j}\, x^j \otimes x^{k-j} \right\rangle \\
&= \sum_{j=0}^{k} \binom{k}{j} \frac{j!\, z^{j-n}}{(j-n)!} \frac{(k-j)!\, w^{k-j-m}}{(k-j-m)!} \\
&= \sum_{j=n}^{k-m} \frac{k!}{(j-n)!(k-j-m)!} z^{j-n} w^{k-j-m} \\
&= \frac{k!}{(k-m-n)!} \sum_{j=n}^{k-m} \frac{(k-m-n)!}{(j-n)!(k-j-m)!} z^{j-n} w^{k-j-m} \\
&= \frac{k!}{(k-m-n)!} (z+w)^{k-m-n} \;=\; \left\langle \mathrm{ev}_{z+w}^{(n+m)}, x^k \right\rangle.
\end{aligned}$$

(ii) Use the Leibnitz formula for derivatives of a product function to get

$$\begin{aligned}
\left\langle \mu^*(\mathrm{ev}_z^{(n)}), x^k \otimes x^l \right\rangle &= \left\langle \mathrm{ev}_z^{(n)}, x^{k+l} \right\rangle \;=\; \sum_{j=0}^{n} \binom{n}{j} \left(\partial_z^j\, z^k\right) \left(\partial_z^{n-j}\, z^l\right) \\
&= \sum_{j=0}^{n} \binom{n}{j} \left\langle \mathrm{ev}_z^{(j)}, x^k \right\rangle \left\langle \mathrm{ev}_z^{(n-j)}, x^l \right\rangle \;=\; \sum_{j=0}^{n} \binom{n}{j} \left\langle \mathrm{ev}_z^{(j)} \otimes \mathrm{ev}_z^{(n-j)}, x^k \otimes x^l \right\rangle.
\end{aligned}$$

(iii) The unit of A° is the adjoint of ϵ

$$\left\langle 1_{A^\circ}, x^k \right\rangle = \epsilon(x^k) \;=\; \delta_{k,0} \;=\; \left\langle \mathrm{ev}_0^{(0)}, x^k \right\rangle.$$

(iv) The counit of A° is the adjoint of η

$$\eta^*(\mathrm{ev}_z^{(n)}) = \left\langle \mathrm{ev}_z^{(n)}, x^0 \right\rangle \;=\; \delta_{n,0}.$$

(v) The antipode of A° is the adjoint of γ

$$\begin{aligned}
\left\langle \gamma^*(\mathrm{ev}_z^{(n)}), x^k \right\rangle &= \left\langle \mathrm{ev}_z^{(n)}, \gamma(x^k) \right\rangle \;=\; \left\langle \mathrm{ev}_z^{(n)}, (-1)^k x^k \right\rangle \\
&= (-1)^k \frac{k!}{(k-n)!} z^{k-n} \;=\; (-1)^n \frac{k!}{(k-n)!} (-z)^{k-n} \;=\; (-1)^n \left\langle \mathrm{ev}_{-z}^{(n)}, x^k \right\rangle.
\end{aligned}$$

All of the formulas coincide with the corresponding ones in $A \otimes \mathbb{C}[\mathbb{C}]$, so we conclude that the linear map $A \otimes \mathbb{C}[\mathbb{C}] \longrightarrow A^\circ$ sending $x^n \otimes v_z \mapsto \mathrm{ev}_z^{(n)}$ is an isomorphism of Hopf algebras.

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for some $m > 0$ and $c_0, c_1, \ldots, c_{m-1} \in \mathbb{C}$. For given m and $c_0, c_1, \ldots, c_{m-1}$ the sequences satisfying this recursion form a m dimensional vector space. If we define the polynomial $P(z) = z^m - \sum_{j=0}^{m-1} c_j z^j$ and assume that its roots are $z_1, \ldots, z_r$ with respective multiplicities $m_1, \ldots, m_r$, then a basis of the space of sequences satisfying the recursion is given by the sequences corresponding to $\mathrm{ev}_{z_j}^{(n)} \in A^\circ$ with $j = 1, 2, \ldots, r$ and $0 \leq n < m_j$. This shows that the restricted dual is spanned by $\left\{\mathrm{ev}_z^{(n)} : (z, n) \in \mathbb{C} \times \mathbb{N}\right\}$. Note also that although the Hopf algebra structure of $\mathbb{C}[\mathbb{Z}]$ is very different, the same analysis works for $\mathbb{C}[\mathbb{Z}]^\circ$, too, with the only difference that z must be non-zero.