Hopf algebras and representations
Spring 2011

Problem sheet 5
Model solutions

Ex 1

(a) Let $a \in A$ be grouplike. By (H2')

$$a = \varepsilon(a)a \qquad \Rightarrow \varepsilon(a) = 1$$

By (H3)

$$S(a)a = \eta(\varepsilon(a)) = 1_A = aS(a)$$

$$\Rightarrow \exists\, a^{-1} \text{ and } S(a) = a^{-1}$$

(b) Let $x \in A$ be primitive. By (H2')

$$x = \varepsilon(x)1_A + \underbrace{\varepsilon(1_A)}_{=1}x \quad \Rightarrow \varepsilon(x)1_A = 0 \Rightarrow \varepsilon(x) = 0$$

By (H3)

$$0 = \eta(\varepsilon(x)) = \mu \circ (id_A \otimes S)(x \otimes 1_A + 1_A \otimes x)$$

$$= x\underbrace{S(1_A)}_{=1} + 1_A S(x)$$

$$\Rightarrow S(x) = -x$$

Ex 2

$(A, \mu, \Delta, \eta, \varepsilon)$ bialgebra. Let $\mu^{op} = \mu \circ S_{A,A}$ and $\Delta^{cop} = S_{A,A} \circ \Delta$. All the claims follow after we check the axioms in the following stages:

(1.) "(μ^{op}, η) satisfy (H1) and (H2)": By associativity of μ the upper part of (H1) for μ^{op} is $a \otimes b \otimes c \mapsto cba$ which is the same as the lower one. Also (H2) follows easily from unitality of (μ, η).

(2.) "$(\Delta^{cop}, \varepsilon)$ satisfy (H1') and (H2')": By coassociativity of Δ we can write $(\Delta \otimes id_A) \circ \Delta(a) = (id_A \otimes \Delta) \circ \Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ the upper and lower part of (H1') for Δ^{cop} equal to $a \mapsto \sum_{(a)} a_{(3)} \otimes a_{(2)} \otimes a_{(1)}$. Also (H2') follows easily from counitality of (Δ, ε).

(3.) (H6) continues to hold, since it doesn't involve the product or coproduct.

(4.) (H5) for (Δ^{cop}, η) and (H5') for (μ^{op}, ε) follow easily since $\mathbb{C}$ is commutative.

(5.) "H4 holds for (μ^{op}, Δ)": (H4) at $a \otimes b$ is

upper $= \Delta(ba)$

lower $= \mu^{op} \otimes \mu^{op} \left(\sum_{(a),(b)} a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)} \right) = \sum_{(a)(b)} (b_{(1)} a_{(1)}) \otimes (b_{(2)} a_{(2)})$

Hence upper = lower $\Leftrightarrow$ (H4) for (μ, Δ) at $b \otimes a$

(6.) "H4 holds for (μ, Δ^{cop})": (H4) at $a \otimes b$ is

upper $= \Delta^{cop}(ab) = S_{A,A}(\Delta(ab))$

lower $= \mu \otimes \mu \left(\sum_{(a),(b)} a_{(2)} \otimes b_{(2)} \otimes a_{(1)} \otimes b_{(1)} \right) = \sum_{(a),(b)} (a_{(2)} b_{(2)}) \otimes (a_{(1)} b_{(1)})$

$= S_{A,A} \left(\sum_{(a),(b)} (a_{(1)} b_{(1)}) \otimes (a_{(2)} b_{(2)}) \right)$

Hence upper = lower $\Leftrightarrow$ (H4) for (μ, Δ) at $a \otimes b$.

(7.) "(H4) holds for (μ^{op}, Δ^{cop})": (H4) at $a \otimes b$ is

upper $= S_{A,A}(\Delta(ba))$

lower $= \mu^{op} \otimes \mu^{op} \left(\sum_{(a),(b)} a_{(2)} \otimes b_{(2)} \otimes a_{(1)} \otimes b_{(1)} \right)$

$= \sum_{(a),(b)} (b_{(2)} a_{(2)}) \otimes (b_{(1)} a_{(1)}) = S_{A,A}\left(\sum_{(a),(b)} (b_{(1)} a_{(1)}) \otimes (b_{(2)} a_{(2)}\right.$

Hence upper = lower $\iff$ (H4) for (μ, Δ) at $b \otimes a$.

Ex 3

(a) By Ex 2, $(A, \mu^{op}, \Delta^{cop}, \eta, \varepsilon)$ is a bialgebra. Hence it is enough to check (H3). Since $S_{A,A} \circ (f \otimes g) \circ S_{A,A} = g \otimes f$ for any linear maps $f, g : A \to A$, the top part of (H3)

$$\begin{tikzcd} A\otimes A \arrow[r, "id_A \otimes \mathcal{S}"] & A\otimes A \arrow[d, "\mu"] \\ A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "\eta"] & A \end{tikzcd} \implies \begin{tikzcd} A\otimes A \arrow[r, "\mathcal{S}\otimes id_A"] & A\otimes A \arrow[d, "\mu^{op}"] \\ A \arrow[u, "\Delta^{op}"] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "\eta"] & A \end{tikzcd}$$

i.e. the lower part of (H3) holds for $(A, \mu^{op}, \Delta^{op}, \eta, \varepsilon)$. Similarly the lower part of (H3) implies the upper part for the opp. co-opp. Hopf algebra.

(b) Claim is that the following are equivalent:

(i) A^{op} admits antipode $\tilde{\mathcal{S}}$ (ii) A^{cop} admits antipode $\tilde{\mathcal{S}}$

(iii) $\mathcal{S}$ is invertible and $\mathcal{S}^{-1} = \tilde{\mathcal{S}}$

"(i) $\Rightarrow$ (iii)": By (H3) in A^{op}, $\overbrace{\sum_{(a)} \tilde{\mathcal{S}}(a_{(2)})\, a_{(1)} = \varepsilon(a)\eta(1)}^{(*)} =$
$= \sum_{(a)} a_{(2)}\, \tilde{\mathcal{S}}(a_{(1)})$. In the convolution algebra,

$\mathcal{S} * id_A = 1_* = id_A * \mathcal{S}$ where 1_* is the unit of the convolution algebra, which was identified as $1_* : a \mapsto \varepsilon(a)\eta(1)$. Since the convolutive inverse is unique, we will show that $\mathcal{S}(\tilde{\mathcal{S}}) * \mathcal{S} = 1_*$.

$$(\mathcal{S}(\tilde{\mathcal{S}}) * \mathcal{S})(a) = \sum_{(a)} \mathcal{S}(\tilde{\mathcal{S}}(a_{(1)}))\,\mathcal{S}(a_{(2)})$$

$$\overset{\mathcal{S} \text{ is homom. of algebras } A \to A^{op}}{=} \sum_{(a)} \mathcal{S}\left(a_{(2)}\, \tilde{\mathcal{S}}(a_{(1)})\right) = \mathcal{S}\left(\sum_{(a)} a_{(2)}\, \tilde{\mathcal{S}}(a_{(1)})\right)$$

$$\overset{(*)}{=} \varepsilon(a)\eta(1)$$

Therefore $\mathcal{S}(\tilde{\mathcal{S}}) = id_A$. Similarly $\tilde{\mathcal{S}}(\mathcal{S}) = id_A$ (For example, exchange the roles of $\mathcal{S}$ and $\tilde{\mathcal{S}}$.)

"(iii) $\Rightarrow$ (ii)": We show that $\tilde{\mathcal{S}} = \mathcal{S}^{-1}$ is an antipode for A^{cop}.

$$\mu \circ (id_A \otimes \mathcal{S}^{-1}) \circ \Delta^{cop}(a) = \mu \Big(\sum_{(a)} a_{(2)} \otimes \mathcal{S}^{-1}(a_{(1)}) \Big)$$

$$= \sum_{(a)} a_{(2)} \mathcal{S}^{-1}(a_{(1)}) = \mathcal{S}^{-1} \circ \mathcal{S} \Big(\sum_{(a)} a_{(2)} \mathcal{S}^{-1}(a_{(1)}) \Big)$$

$$\overset{\mathcal{S} \text{ is antimorphism}}{=} \mathcal{S}^{-1} \Big(\sum_{(a)} a_{(1)} \mathcal{S}(a_{(2)}) \Big) = \mathcal{S}^{-1}(\varepsilon(a) \eta(1)) = \varepsilon(a) \eta(1)$$

Similarly the other half of (HS).

"(ii) $\Rightarrow$ (i)" Since for any linear maps $f, g : A \to A$ it holds that $S_{A,A} \circ (f \otimes g) \circ S_{A,A} = g \otimes f$, we have

$$\begin{tikzcd} A \otimes A \arrow[r, "id_A \otimes \tilde{\mathcal{S}}"] & A \otimes A \arrow[d, "\mu"] \\ A \arrow[u, "\Delta^{cop}"] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "\eta"] & A \end{tikzcd} \iff \begin{tikzcd} A \otimes A \arrow[r, "\tilde{\mathcal{S}} \otimes id_A"] & A \otimes A \arrow[d, "\mu^{op}"] \\ A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "\eta"] & A \end{tikzcd}$$

Similarly the other half.

Ex 4

Consider the subspace $X \subset B$ of those elements $x \in B$ that satisfy

$$(\mu \circ (\wp \otimes id_B) \circ \Delta)(x) = \varepsilon(x) 1_B = (\mu \circ (id_B \otimes \wp) \circ \Delta)(x)$$

Then $1 \in X$, since $\Delta(1) = 1 \otimes 1$ and $\wp(1) = 1$ (by assumptions $\wp : B \to B$ is homom. of algebras from B to B^{op}), and X is closed under the product: let $x, y \in X$

$$\mu \circ (\wp \otimes id_B) \circ \Delta(xy) \overset{(H4)}{=} \mu \circ (\wp \otimes id_B)\Big(\sum_{(x),(y)} (x_{(1)} y_{(1)}) \otimes (x_{(2)} y_{(2)})\Big)$$

$$\overset{\substack{\wp(ab) \\ = \wp(b)\wp(a)}}{=} \sum_{(x)(y)} \wp(y_{(1)}) \wp(x_{(1)}) x_{(2)} y_{(2)}$$

$$= \sum_{(y)} \wp(y_{(1)}) \Big[\sum_{(x)} \wp(x_{(1)}) x_{(2)} \Big] y_{(2)}$$

$$\overset{x \in X}{=} \varepsilon(x) \sum_{(y)} \wp(y_{(1)}) y_{(2)} \overset{y \in X}{=} \varepsilon(x)\varepsilon(y) 1_B \overset{(H5')}{=} \varepsilon(xy) 1_B$$

Similarly $\mu \circ (id_B \otimes \wp) \circ \Delta(xy) = \varepsilon(xy) 1_B$. Hence $xy \in X$.

Therefore $X \subset B$ is a subalgebra. Since it contains the generators, $X = B$.

Ex 5

(a) If there exists a bialgebra structure on $H = H_q$ satisfying $\Delta(a) = a \otimes a$ and $\Delta(b) = a \otimes b + b \otimes 1$, then Δ and ε are homom. of algebras

$$\Rightarrow \Delta(1) = 1 \otimes 1 \quad , \quad \varepsilon(1) = 1$$

From $a^{-1}a = 1 = aa^{-1}$, it follows that $\Delta(a) \in H \otimes H$ is invertible and $\Delta(a^{-1}) = \Delta(a)^{-1} = (a \otimes a)^{-1} = a^{-1} \otimes a^{-1}$.

Since a and a^{-1} are grouplike, $\varepsilon(a) = 1 = \varepsilon(a^{-1})$.

By (H2')

$$b = \underbrace{\varepsilon(a)}_{=1} b + \varepsilon(b) 1 \quad \Rightarrow \quad \varepsilon(b) = 0$$

Therefore we have to have

$$\Delta(a) = a \otimes a \qquad \varepsilon(a) = 1$$

$$\Delta(a^{-1}) = a^{-1} \otimes a^{-1} \qquad \varepsilon(a^{-1}) = 1$$

$$\Delta(b) = a \otimes b + b \otimes 1 \qquad \varepsilon(b) = 0$$

We can show they satisfy the axioms (H1') and (H2'). If we assume that Δ and ε are homom. of algebras, i.e. satisfy (H4-6), then we can prove similarly as in Ex 4 that the whole algebra satisfies (H1') and (H2'). Bialgebra structure is unique, since it is unique for the generators.

(b) If require (H3) for S, then $S(a) = a^{-1}$ and $S(a^{-1}) = a$, since they are grouplike.

$$\Rightarrow \quad 0 = \eta(\varepsilon(b)) = \mu \circ (id_H \otimes S) \circ \Delta(b)$$

$$= a S(b) + b \underbrace{S(1)}_{=1,\ S \text{ is antimorphism}}$$

$$\Rightarrow \quad S(b) = -a^{-1}b = -q^{-1} b a^{-1}$$

By Ex 4, this and antimorphism property defines a unique Hopf algebra structure on H.

(c) $\varepsilon(b^m a^n) = \varepsilon(b^m)\,\varepsilon(a^n) = \begin{cases} 0, & \text{if } m>0 \\ 1, & \text{if } m=0 \end{cases}$

$$S(a^n) = S(a)^n = a^{-n}$$

$m>0$, $S(b^m a^n) = S(a)^n S(b)^m$

commute twice

commute once

$$= (-1)^m a^{-n} \underbrace{a^{-1} b\, a^{-1} b \cdots a^{-1} b\, a^{-1} b}_{m \text{ times}}$$

$$= (-1)^m q^{-(1+2+3+\ldots+m)} \underbrace{a^{-n} b^m}_{= q^{-nm} b^m a^{-n}} a^{-m}$$

$$= (-1)^m q^{-\left[\frac{m(m+1)}{2} + nm\right]} b^m a^{-n-m}$$

(d) $x = a \otimes b$, $y = b \otimes 1$

$$\Rightarrow xy = (ab) \otimes b = q\,(ba) \otimes b = q\,yx$$

$$\Rightarrow \Delta(b^m) = \Delta(b)^m = (x+y)^m \text{, use the hint}$$

$$= \sum_{k=0}^{m} \left[\!\left[\begin{matrix} m \\ k \end{matrix} \right]\!\right]_q y^k x^{m-k}$$

$$= \sum_{k=0}^{m} \left[\!\left[\begin{matrix} m \\ k \end{matrix} \right]\!\right]_q (b^k a^{m-k}) \otimes b^{m-k}$$

$$\Rightarrow \Delta(b^m a^n) = \sum_{k=0}^{m} \left[\!\left[\begin{matrix} m \\ k \end{matrix} \right]\!\right]_q (b^k a^{n+m-k}) \otimes (b^{m-k} a^n)$$