Hopf algebras and representations Problem sheet 4
Spring 2011 Model solutions

## Ex1

(a) We prove the claim by induction. When $n=1$

$$(a+b)^1 = \sum_{k=0}^{1} \left[\!\!\left[\begin{matrix}1\\k\end{matrix}\right]\!\!\right]_q b^{1-k} a^k$$

because $\left[\!\!\left[\begin{matrix}1\\0\end{matrix}\right]\!\!\right]_q = \left[\!\!\left[\begin{matrix}1\\1\end{matrix}\right]\!\!\right]_q = 1$. Suppose now that

$$(a+b)^n = \sum_{k=0}^{n} \left[\!\!\left[\begin{matrix}n\\k\end{matrix}\right]\!\!\right]_q b^{n-k} a^k$$

Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b) \sum_{k=0}^{n} \left[\!\!\left[\begin{matrix}n\\k\end{matrix}\right]\!\!\right]_q b^{n-k} a^k \\
&= a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \left\{ \left[\!\!\left[\begin{matrix}n\\k-1\end{matrix}\right]\!\!\right]_q \underbrace{a b^{n-k+1} a^{k-1}}_{=q^{n-k+1} b^{n-k+1} a^k} + \left[\!\!\left[\begin{matrix}n\\k\end{matrix}\right]\!\!\right] b^{n-k+1} a^k \right\} \\
&= \sum_{k=0}^{n+1} \left[\!\!\left[\begin{matrix}n+1\\k\end{matrix}\right]\!\!\right]_q b^{n-k+1} a^k
\end{aligned}$$

where we used $\left[\!\!\left[\begin{matrix}n\\0\end{matrix}\right]\!\!\right]_q = \left[\!\!\left[\begin{matrix}n\\n\end{matrix}\right]\!\!\right]_q = 1$ and

$$\begin{aligned}
&\left[\!\!\left[\begin{matrix}n\\k-1\end{matrix}\right]\!\!\right] q^{n-k+1} + \left[\!\!\left[\begin{matrix}n\\k\end{matrix}\right]\!\!\right]_q \\
&= \frac{\prod_{j=1}^{n} (1-q^j)}{\prod_{j=1}^{k} (1-q^j) \prod_{j=1}^{n-k+1} (1-q^j)} \cdot \underbrace{\left[ (1-q^k) \cdot q^{n-k+1} + (1-q^{n-k+1}) \right]}_{= 1 - q^{n+1}} \\
&= \left[\!\!\left[\begin{matrix}n+1\\k\end{matrix}\right]\!\!\right]_q
\end{aligned}$$

(b) $q = e^{i2\pi/n}$ is a zero of numerator of $\left[\!\!\left[\begin{matrix}n\\k\end{matrix}\right]\!\!\right]_q$ for all $k$, but the denominator is non-zero for any $1 \leq k \leq n-1$. Hence $\left[\!\!\left[\begin{matrix}n\\k\end{matrix}\right]\!\!\right]_q = 0$ for $1 \leq k \leq n-1$ and the claim follows.

Ex2

$C$ has basis $\{c, s\}$. $\Delta(c) = c \otimes c - s \otimes s$, $\Delta(s) = c \otimes s + s \otimes c$

Suppose that there is counit $\varepsilon : C \to \mathbb{C}$. Then by (H2')

$$c = (\varepsilon \otimes id_C) \circ \Delta(c) = \varepsilon(c)c - \varepsilon(s)s$$

$$\Rightarrow \quad \varepsilon(c) = 1 \quad , \quad \varepsilon(s) = 0$$

Now we can show that linear extension of this $\varepsilon$ together with above $\Delta$ satisfy the axioms (H1') and (H2').

## Ex 3

Assume that $(B, \mu, \eta)$ is an algebra and $(B, \Delta, \varepsilon)$ is a bialgebra. Denote $\mu_B = \mu$, $\eta_B = \eta$, ..., and by $\mu_{\mathbb{C}}$, $\Delta_{\mathbb{C}}$, ... the structure on $\mathbb{C}$ and by $\mu_{B\otimes B}$, $\Delta_{B\otimes B}$, ... the structure on $B \otimes B$.

The statement that $\Delta$ and $\varepsilon$ are homom. of algebras means

$$\begin{cases} \Delta \circ \mu_B = \mu_{B\otimes B} \circ (\Delta \otimes \Delta) \\ \Delta \circ \eta_B = \eta_{B\otimes B} \\ \varepsilon \circ \mu_B = \mu_{\mathbb{C}} \circ (\varepsilon \otimes \varepsilon) \\ \varepsilon \circ \eta_B = \eta_{\mathbb{C}} \end{cases}$$

Since $\mu_{B\otimes B} = (\mu_B \otimes \mu_B) \circ (id_B \otimes S_{B,B} \otimes id_B)$, $\eta_{B\otimes B} = \eta_B \otimes \eta_B$, $\mu_{\mathbb{C}} = id_{\mathbb{C}}$ [Note $\mu_{\mathbb{C}}(a \otimes b) = ab = a \otimes b$] and $\eta_{\mathbb{C}} = id_{\mathbb{C}}$, then "$\Delta$ is homom." $\Leftrightarrow$ "(H4) and (H5)" and "$\varepsilon$ is homom." $\Leftrightarrow$ "(H5') and (H6)"

Similarly the statement that $\mu$ and $\eta$ are homom. of coalgebras means

$$\begin{cases} \Delta_B \circ \mu = (\mu \otimes \mu) \circ \Delta_{B\otimes B} \\ \varepsilon_B \circ \mu = \varepsilon_{B\otimes B} \\ \Delta_B \circ \eta = (\eta \otimes \eta) \circ \Delta_{\mathbb{C}} \\ \varepsilon_B \circ \eta = \varepsilon_{\mathbb{C}} \end{cases}$$

Since $\Delta_{B\otimes B} = (id_B \otimes S_{B,B} \otimes id_B) \circ (\Delta_B \otimes \Delta_B)$, $\varepsilon_{B\otimes B} = \varepsilon_B \otimes \varepsilon_B$, $\Delta_{\mathbb{C}} = id_{\mathbb{C}} \otimes id_{\mathbb{C}}$, $\varepsilon_{\mathbb{C}} = id_{\mathbb{C}}$, then "$\mu$ is homom." $\Leftrightarrow$ "(H4) and (H5')" and "$\eta$ is homom." $\Leftrightarrow$ "(H5) and (H6)"

## Ex 4

(a) Let $(B, \mu, \Delta, \eta, \varepsilon)$ be a two-dimensional bialgebra. Let $1_B = \eta(1)$ which is the unit of $B$. Since $\varepsilon : B \to \mathbb{C}$ is linear map with $\varepsilon(1_B) = 1$ by (H6), the vector space $B = \mathbb{C}1_B \oplus \operatorname{Ker}(\varepsilon)$. Let $x \in \operatorname{Ker}(\varepsilon)$, $x \neq 0$. Then by (H5')
$\varepsilon(x^2) = \varepsilon(x)^2 = 0$ and hence $x^2 = \lambda x$ for some $\lambda \in \mathbb{C}$.

So far we know everything about $\eta, \mu$, and $\varepsilon$: in basis $\{1_B, x\}$

$$\mu: \begin{array}{c|cc} & 1_B & x \\ \hline 1_B & 1_B & x \\ x & x & \lambda x \end{array} \qquad \begin{array}{l} \varepsilon(1_B) = 1 \\ \varepsilon(x) = 0 \end{array} \qquad (*)$$

By (H5), $\Delta(1_B) = 1_B \otimes 1_B$. Write $\Delta(x)$ as

$$\Delta(x) = s\, 1_B \otimes 1_B + t\, 1_B \otimes x + u\, x \otimes 1_B + v\, x \otimes x$$

where $s, t, u, v \in \mathbb{C}$. Then by (H2')

$$s 1_B + t x = x = s 1_B + u x$$

$$\Rightarrow \quad s = 0, \ t = 1, \ u = 1 \quad \Rightarrow \Delta(x) = 1_B \otimes x + x \otimes 1_B + v\, x \otimes x$$

(H4) evaluated at $x \otimes x$ gives

$$\lambda [1_B \otimes x + x \otimes 1_B + v\, x \otimes x] = \Delta(\lambda x) = \Delta(x^2)$$

$$= \lambda 1_B \otimes x + \lambda x \otimes 1_B + 2 x \otimes x + 4 \lambda v\, x \otimes x + \lambda^2 v^2 x \otimes x$$

$$\Leftrightarrow \quad 2 + 3\lambda v + \lambda^2 v^2 = 0 \quad \Leftrightarrow \quad \lambda v = -2 \ \text{or} \ \lambda v = -1$$

Therefore $\lambda \neq 0$ and we can redefine $x$ so that $\lambda = 1$ and then $v = -2$ or $v = -1$. Those two bialgebras are non-isomorphic. It is now possible to check the axioms with $\eta, \mu, \varepsilon$ as in $(*)$ with $\lambda = 1$ ~~and~~ $\Delta$ given by

$$\Delta(1_B) = 1_B \otimes 1_B \qquad \Delta(x) = 1_B \otimes x + x \otimes 1_B + v\, x \otimes x$$

$v = -1$ or $v = -2$.

(b) Suppose that $\mathcal{S}: B \to B$ is an antipode and $(B, \mu, \Delta, \eta, \varepsilon, \mathcal{S})$ is a Hopf algebra. Apply (H3) to $1_B$

$$\Rightarrow 1_B = \eta(\varepsilon(1_B)) = 1_B\,\mathcal{S}(1_B) = \mathcal{S}(1_B)$$

and to $x$

$$\Rightarrow 0 = \eta(\varepsilon(x)) = \sum_{(x)} x_{(1)}\,\mathcal{S}(x_{(2)}) = 1_B\,\mathcal{S}(x) + x\,\mathcal{S}(1_B) + \nu x\,\mathcal{S}(x)$$

$$\Rightarrow \mathcal{S}(x) = -x - \nu x\,\mathcal{S}(x) \in \mathrm{Ker}(\varepsilon) \quad [\text{Ker is ideal}]$$

$$\Rightarrow \mathcal{S}(x) = \omega x \quad \text{for some } \omega \in \mathbb{C}$$

$$\Rightarrow \omega(1+\nu) = -1 \Rightarrow \nu \neq -1 \Rightarrow \nu = -2,\ \omega = 1$$

Therefore only $\nu = -2$ admits Hopf algebra structures and then $\mathcal{S}(1_B) = 1_B$ and $\mathcal{S}(x) = x$.

Note that $y = 1 - 2x$ satisfies

$$\mu(y, y) = 1_B, \quad \Delta(y) = y \otimes y, \quad \mathcal{S}(y) = y$$

Therefore this Hopf algebra is isomorhic to the group algebra of cyclic group of order 2.