Hopf algebras and representations
Spring 2011

Problem sheet 3
Model solutions

Ex 1

(a) In the exercise 4 of sheet 1, we saw four one-dimensional reprentations and one two-dimensional. They are all the irreducible ones. The character table:

	$\{e\}$	$\{r, r^3\}$	$\{r^2\}$	$\{m, mr^2\}$	$\{mr, mr^3\}$
$r \to 1, m \to 1$	1	1	1	1	1
$r \to -1, m \to 1$	1	-1	1	1	-1
$r \to 1, m \to -1$	1	1	1	-1	-1
$r \to -1, m \to -1$	1	-1	1	-1	1
V	2	0	-2	0	0

two-dimensional

For the last line, compute $\rho(r)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\rho(m)\rho(r) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) By character theory and the above table

$$\chi_{V \otimes V} = (\chi_V)^2 = (4, 0, 4, 0, 0)$$
$$= \sum_{k=1}^{4} \chi_{U_k}$$

where U_k are the 1D representations of the above table.

Ex 2

(a) Let $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ be the dual-basis of $\{e_1, e_2, e_3, e_4\}$

Then

$$\chi_V(\sigma) = \sum_{k=1}^{4} \varphi_k(\sigma . e_k) = \sum_{k=1}^{4} \delta_{k,\sigma(k)}$$

$$= \# \text{ elements fixed by } \sigma$$

The conjugacy classes are given by the cycle decomposition types. They can be therefore represented by

conjugacy classes	e	(12)	(123)	(1234)	$(12)(34)$
no. elements	1	6	8	6	3

In this order, the character of V is

$$\chi_V = (4, 2, 1, 0, 0)$$

(b) $V'' = \operatorname{span}(\{e_1 + e_2 + e_3 + e_4\})$ is clearly trivial as representation. Write $V = V' \oplus V''$ where V' is ~~a subrepresentation~~ complementary to V''. By character theory

$$\chi_{V'} = \chi_V - \chi_{V''} = (3, 1, 0, -1, -1)$$

Now $\frac{1}{|S_4|} \sum_g |\chi_{V'}(g)|^2 = \frac{1}{24} (1 \cdot 3^2 + 6 \cdot 1^2 + 8 \cdot 0 + 6 \cdot (-1)^2 + 3 \cdot (-1)^2)$

$= \frac{1}{24}(9 + 6 + 6 + 3) = 1$ and hence V' is irreducible.

(c) By the example from lectures, S_4 has five irreducible reps. and their dimensions are $1, 1, 2, 3, 3$.

One-dimensionals are $U_1 =$ trivial, $U_2 =$ alternating and in (b) we found $U_4 = V'$ which is three-dimensional.

- By character theory, $U_5 = U_4 \otimes U_4$ has character

$$\chi_{U_5} = \chi_{U_1}\chi_{U_4} = (3, -1, 0, 1, -1)$$

Now since $\frac{1}{|S_4|}\sum_g |\chi_{U_5}|^2 = \frac{1}{|S_4|}\sum_g |\chi_{U_4}|^2 = 1$, U_5 is irreducible. Missing character of a two-dimension U_3 can be solved by orthogonality.

	e	(12)	(123)	(1234)	$(12)(34)$
U_1	1	1	1	1	1
U_2	1	-1	1	-1	1
U_3	2	0	-1	0	2
U_4	3	1	0	-1	-1
U_5	3	-1	0	1	-1

Ex 3

The character table of S_3 is

	e	(12)	(123)
U = trivial	1	1	1
U' = alternating	1	-1	1
V	2	0	-1

For the last line remember that in a suitable basis

$$\rho((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho((123)) = \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$$

where $\omega = e^{i\frac{2\pi}{3}}$. Write

$$V^{\otimes n} = a_n U \oplus b_n U' \oplus c_n V$$

By character theory,

$$\chi_{V^{\otimes n}} = (\chi_V)^n = (2^n, 0, (-1)^n)$$

$$\Rightarrow \begin{cases} a_n + b_n + 2c_n = 2^n \\ a_n - b_n = 0 \\ a_n + b_n - c_n = (-1)^n \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = b_n = 2^{n-1} - c_n \\ c_n = \frac{1}{3}(2^n - (-1)^n) \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = b_n = \frac{1}{3}(2^{n-1} - (-1)^{n-1}) \\ c_n = \frac{1}{3}(2^n - (-1)^n) \end{cases}$$

Ex 4

(a) Let $a = \sum_g \alpha(g) e_g \in A = \mathbb{C}[G]$. Then

$$[a \in Z] \iff [az = za \quad \forall z \in A] \iff [a e_g = e_g a \quad \forall g \in G]$$

by linearity of the formula $az = za$ with respect to z. Therefore $a \in Z$, if and only if for each $h \in G$

$$\sum_g \alpha(g) e_g = a = e_{h^{-1}} a e_h$$

$$= e_{h^{-1}} \Big[\sum_g \alpha(g) e_g \Big] e_h$$

$$= \sum_g \alpha(g) e_{h^{-1} g h}, \quad \text{change of variables } g' = h^{-1} g h$$

$$= \sum_{g'} \alpha(h g' h^{-1}) e_{g'}$$

i.e. $\alpha(g) = \alpha(hgh^{-1}) \quad \forall g, \forall h.$

(b) If α is constant on conjugacy classes, then $a = \sum_g \alpha(g) e_g$ is in Z of $A = \mathbb{C}[G]$ by the previous proof. Hence $a_V = \sum_g \alpha(g) \rho(g) \in \mathrm{End}(V)$ commutes with any $\rho(h)$: $a_V \rho(h) = \rho(h) a_V$. Here $\rho: G \to GL(V)$ is an representation. If V is irreducible, then by Schur's lemma $a_V = \lambda \, \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.

$$\Rightarrow \lambda \dim(V) = tr(a_V) = \sum_g \alpha(g)\, tr(\rho(g))$$

$$= |G| \, (\chi_{V^*}, \alpha) = 0$$

since also V^* is an irreducible rep.: $\frac{1}{|G|} \sum_g |\chi_{V^*}(g)|^2 = \frac{1}{|G|} \sum_g |\chi_V(g)|^2 = 1$. Therefore $\lambda = 0$. Now in any V, also $a_V = 0$.

Take V the standard rep. spanned by $\{e_g : g \in G\}$. Then $\rho(g) = e_g$ as elements of $\mathrm{End}(V)$ are lin. independent. $a_V = 0 \Rightarrow \alpha(g) = 0 \ \forall g$

(c) The space of class functions (functions on conjugacy classes) is a vector space with the dimension equal to the number of conjugacy classes. In (b) we proved that there is no function that is orthogonal to all the characters of irreducible representations. Hence the characters of irreducibles have to span the whole space. Especially there are at least the number of conjugacy classes non-isomorphic irreducible representations.