Problem sheet 9

Exercise 1: *A central element in braided Hopf algebras*
Let A be a Hopf algebra such that the antipode $\gamma : A \to A$ has an inverse $\gamma^{-1} : A \to A$. In the lectures we found an invertible element $u \in A$, which satisfies $\gamma(\gamma(x)) = u\,x\,u^{-1}$ for all $x \in A$.

Show that $\gamma(u)u$ is in the center of A, i.e. $\gamma(u)ux = x\gamma(u)u$ for all $x \in A$, and that $\gamma(u)\,u = u\,\gamma(u)$.

Exercise 2: *Inverse of the antipode for the building block of quantum groups*
Let $q \in \mathbb{C} \setminus \{0\}$ and let H_q be the Hopf algebra, which as an algebra is generated by a, a^{-1}, b with relations $aa^{-1} = 1$, $a^{-1}a = 1$, $ab = q\,ba$, and has $\Delta(a) = a \otimes a$, $\Delta(b) = a \otimes b + b \otimes 1$.

Show that the antipode γ of H_q is invertible with inverse given by

$$\gamma^{-1}(b^m a^n) \;=\; (-1)^m \, q^{-\frac{1}{2}m(m-1)-mn} \, b^m a^{-m-n}.$$

Exercise 3: *Universal R-matrix for Drinfeld double, the rest of the proof*
Let A be a finite dimensional Hopf algebra (with invertible antipode), and let A^* be its dual Hopf algebra and $\mathcal{D}(A)$ the Drinfeld double of A. Let $(e_i)_{i=1}^d$ be a basis of A, and $(\delta^i)_{i=1}^d$ the dual basis of A^*. We claim that

$$R \;=\; (\iota_A \otimes \iota_{A^*})(\mathrm{coev}(1)) \;=\; \sum_{i=1}^{d} (e_i \otimes 1^*) \otimes (1 \otimes \delta^i)$$

is a universal R-matrix for $\mathcal{D}(A)$. In the lectures we (almost) verified the properties (R0) and (R1) for R and $\mathcal{D}(A)$.

Verify the properties (R2) and (R3) for R and $\mathcal{D}(A)$.

Exercise 4: *The Drinfeld double of the Hopf algebra of a finite group*
Let G be a finite group, and $A = \mathbb{C}[G]$ the Hopf algebra of the group G. Let $(e_g)_{g \in G}$ be the natural basis of A, and let $(f_g)_{g \in G}$ be the dual basis of A^*.

(a) Show that A^* is, as an algebra, isomorphic to the algebra of complex valued functions on G with pointwise multiplication: when $\phi, \psi : G \to \mathbb{C}$, the product $\phi\psi$ is the function defined by $(\phi\psi)(g) = \phi(g)\psi(g)$ for all $g \in G$.

(b) Find explicit formulas for the coproduct, counit and antipode of A^* in the basis $(f_g)_{g \in G}$.

Let $\mathcal{D}(A)$ be the Drinfeld double of A.

(c) Find explicit formulas for the coproduct, counit and unit of $\mathcal{D}(A)$ in the basis $(e_h \otimes f_g)_{g,h \in G}$.

(d) Show that the product $\mu_{\mathcal{D}}$ and antipode $\gamma_{\mathcal{D}}$ of $\mathcal{D}(A)$ are given by the following formulas

$$\begin{aligned} \mu_{\mathcal{D}}\Big((e_{h'} \otimes f_{g'}) \otimes (e_h \otimes f_g)\Big) &= \delta_{g', hgh^{-1}} \, e_{h'h} \otimes f_g \\ \gamma_{\mathcal{D}}(e_h \otimes f_g) &= e_h \otimes f_{hg^{-1}h^{-1}}. \end{aligned}$$