Problem sheet 4

Exercise 1: *The q-binomial formula*
Define the q-binomial coefficient, for $n \in \mathbb{N}$, $0 \leq k \leq n$, as the following rational function of q

$$\left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right] = \frac{\prod_{j=1}^{n}(1-q^j)}{\prod_{j=1}^{k}(1-q^j)\ \prod_{j=1}^{n-k}(1-q^j)}.$$

Now let $q \in \mathbb{C}$, and denote the value of that rational function at q by

$$\left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right]_q .$$

Suppose A is an algebra and $a, b \in A$ are two elements which satisfy the relation

$$a\, b = q\, b\, a.$$

(a) Show that for any $n \in \mathbb{N}$ we have

$$(a+b)^n = \sum_{k=0}^{n} \left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right]_q b^{n-k}\, a^k.$$

(*Hint:* How would you prove the ordinary binomial formula?)

(b) If $q = e^{\mathrm{i}2\pi/n}$, show that $(a+b)^n = a^n + b^n$.

Exercise 2: *A coalgebra from trigonometric addition formulas*
Let C be a vector space with basis $\{c, s\}$. Define $\Delta : C \to C \otimes C$ by linear extension of

$$c \mapsto c \otimes c - s \otimes s \quad , \qquad s \mapsto c \otimes s + s \otimes c.$$

Does there exist $\epsilon : C \to \mathbb{C}$ such that (C, Δ, ϵ) becomes a coalgebra?

Definitions for the last two exercises:

- Let (C, Δ, ϵ) and (C', Δ', ϵ') be two coalgebras. A linear map $f : C \to C'$ is called a homomorphism of coalgebras if for all $a \in C$, with Sweedler's notation for the coproduct $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \in C \otimes C$, we have
$$\Delta'\big(f(a)\big) = \sum_{(a)} f(a_{(1)}) \otimes f(a_{(2)}) \qquad \text{and} \qquad \epsilon'\big(f(a)\big) = \epsilon(a).$$
- Let $(B, \mu, \Delta, \eta, \epsilon)$ and $(B', \mu', \Delta', \eta', \epsilon')$ be two bialgebras. A linear map $f : B \to B'$ is called a homomorphism of bialgebras if f is a homomorphism of algebras from (B, μ, η) to (B', μ', η') and a homomorphism of coalgebras from (B, Δ, ϵ) to (B', Δ', ϵ').

Exercise 3: *Alternative definitions of bialgebra*
Let B be a vector space and suppose that
$$\mu : B \otimes B \to B \qquad\qquad \eta : \mathbb{C} \to B$$
$$\Delta : B \to B \otimes B \qquad\qquad \epsilon : B \to \mathbb{C}$$
are linear maps such that (B, μ, η) is an algebra and (B, Δ, ϵ) is a coalgebra.

Show that the following conditions are equivalent:

(i) Both Δ and ϵ are homomorphisms of algebras.

(ii) Both μ and η are homomorphisms of coalgebras.

(iii) $(B, \mu, \Delta, \eta, \epsilon)$ is a bialgebra.

"Clarifications": The algebra structure on $\mathbb{C}$ is using the product of complex numbers. The coalgebra structure on $\mathbb{C}$ is such that the coproduct and counit are both identity maps of $\mathbb{C}$ (for coproduct identify $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$, and for the counit note that $\mathbb{C}$ itself is the ground field). The algebra structure on $B \otimes B$ is the tensor product of two copies of the algebra B, i.e. with the product determined by $(b' \otimes b'')\,(b''' \otimes b'''') = b'b''' \otimes b''b''''$. The coalgebra structure in $B \otimes B$ is the tensor product of two copies of the coalgebra B, i.e. when $\Delta(b') = \sum b'_{(1)} \otimes b'_{(2)}$ and $\Delta(b'') = \sum b''_{(1)} \otimes b''_{(2)}$ then the coproduct of $b' \otimes b''$ is $\sum \big(b'_{(1)} \otimes b''_{(1)}\big) \otimes \big(b'_{(2)} \otimes b''_{(2)}\big)$ and counit is simply $b' \otimes b'' \mapsto \epsilon(b')\,\epsilon(b'')$.

Exercise 4: *Two dimensional bialgebras*

(a) Classify all two-dimensional bialgebras up to isomorphism.

(b) Which of the two-dimensional bialgebras admit a Hopf algebra structure?