Problem sheet 13

Ribbon Hopf algebras:

Let A be a braided Hopf algebra with universal R-matrix $R \in A \otimes A$, and denote $R_{21} = S_{A,A}(R)$. Assume that there exists a central element $\theta \in A$ such that

$$\Delta(\theta) = (R_{21}\, R)^{-1}\, (\theta \otimes \theta) \qquad , \qquad \epsilon(\theta) = 1 \qquad \text{and} \qquad \gamma(\theta) = \theta.$$

Then A is said to be *ribbon Hopf algebra* and θ is called *ribbon element*.

Exercise 1: *Twists in modules over ribbon Hopf algebras*

Assume that A is a ribbon Hopf algebra and denote the braiding of A-modules V and W by $c_{V,W}$.

(a) Show that the ribbon element θ is invertible.

(b) For any A-module V define a linear map $\Theta_V : V \to V$ by $\Theta_V(v) = \theta^{-1}.v$ for all $v \in V$. Prove the following:

- When $f : V \to W$ is an A-module map, we have $\Theta_W \circ f = f \circ \Theta_V$.
- When V is an A-module, and V^* is the dual A-module we have $\Theta_{V^*} = (\Theta_V)^*$ (the right hand side is the transpose of Θ_V).
- When V and W are A-modules, we have $\Theta_{V \otimes W} = (\Theta_V \otimes \Theta_W) \circ c_{W,V} \circ c_{V,W}$.

Exercise 2: *Properties of the element u in braided Hopf algebras*

Recall that when A is a braided Hopf algebra with universal R-matrix $R \in A \otimes A$, we set

$$u = \Big(\mu \circ (\gamma \otimes \mathrm{id}_A)\Big)\, (R_{21}).$$

We have seen that $\gamma(\gamma(x)) = u\, x\, u^{-1}$ for all $x \in A$ and that $u\, \gamma(u) = \gamma(u)\, u$ is central in A.

(a) Show that $\epsilon(u) = 1$.

(b) Show that $R_{21}\, R\, \Delta(x) = \Delta(x)\, R_{21}\, R$ for all $x \in A$.

(c) Show that

$$\Delta(u) = (R_{21}\, R)^{-1}\, (u \otimes u).$$

Hint: This requires the use of almost all the properties of the universal R-matrix that we have seen!

(d) Show that $u\, \gamma(u^{-1})$ is grouplike.

Exercise 3: *The center of* $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$
Let q be a root of unity, and assume that the smallest positive integer e such that $q^e \in \{\pm 1\}$ is odd and satisfies $q^e = +1$. Let $A = \widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$ be the quotient of $\mathcal{U}_q(\mathfrak{sl}_2)$ by the relations $E^e = 0$, $F^e = 0$, $K^e = 1$ (see *Problem sheet 10: Exercise 3*). A basis of A is

$$E^a\, F^b\, K^c \qquad \text{with} \qquad a, b, c \in \{0, 1, 2, \ldots, e-1\}\,.$$

(a) Show that the center of A is e-dimensional and a basis of the center is $1, C, C^2, C^3, \ldots, C^{e-1}$, where C is the quadratic Casimir

$$C \;=\; E\,F + \frac{1}{(q-q^{-1})^2}\left(q^{-1}\,K + q\,K^{-1}\right).$$

Hint: This can be done in different ways, but one possible strategy is the following:

- Describe the subspace of elements commuting with K.
- Write down the condition for elements to commute with both K and F, using the formulas of *Problem sheet 10: Exercise 3*, and from this argue that the dimension of the center is at most e.
- Argue that the powers of C are linearly independent central elements.

(b) Show that the unit is the only grouplike central element in $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$.

Exercise 4: *The Hopf algebra* $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$ *is ribbon*
Let q be a root of unity, and assume that the smallest positive integer e such that $q^e \in \{\pm 1\}$ is odd and satisfies $q^e = +1$. Then

$$R \;=\; \frac{1}{e}\sum_{i,j,k=0}^{e-1} \frac{(q-q^{-1})^k}{[k]_q!}\, q^{k(k-1)/2+2k(i-j)-2ij}\; E^k K^i \otimes F^k K^j$$

is a universal R-matrix for $A = \widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$ (see also *Problem sheet 12: Exercise 2*).

(a) Show that K commutes with $u = (\mu \circ (\gamma \otimes \mathrm{id}_A))\,(R_{21})$.

(b) Show that $\gamma(\gamma(x)) = K\,x\,K^{-1}$ for all $x \in A$. Recalling a similar property of u, show that $K^{-1}\,u$ is a central element.

(c) Show that $K^{-2}\,u\,\gamma(u^{-1})$ is a grouplike central element. Conclude that $\gamma(K^{-1}u) = K^{-1}u$.

(d) Show that $\theta = K^{-1}\,u$ is a ribbon element.

Hint: Use the results of the previous exercises!