Problem sheet 10

Exercise 1: *A central element in D_q*
Let q be a non-zero complex number which is not a root of unity, and let $D_q = \mathcal{D}(H_q, H_q')$ be the Hopf algebra which as an algebra is generated by $\alpha, \alpha^{-1}, \beta, \tilde{\alpha}, \tilde{\alpha}^{-1}, \tilde{\beta}$ with relations

$$\begin{aligned} \alpha\alpha^{-1} &= 1 = \alpha^{-1}\alpha & \tilde{\alpha}\tilde{\alpha}^{-1} &= 1 = \tilde{\alpha}^{-1}\tilde{\alpha} \\ \alpha\beta &= q\,\beta\alpha & \tilde{\alpha}\tilde{\beta} &= q\,\tilde{\beta}\tilde{\alpha} \\ \alpha\tilde{\beta} &= q^{-1}\,\tilde{\beta}\alpha & \tilde{\alpha}\beta &= q^{-1}\,\beta\tilde{\alpha} \\ \alpha\tilde{\alpha} &= \tilde{\alpha}\alpha & \tilde{\beta}\beta - \beta\tilde{\beta} &= \alpha - \tilde{\alpha}. \end{aligned}$$

Consider an element of the form

$$\nu = \tilde{\beta}\beta + r\,\alpha + s\,\tilde{\alpha}.$$

Find values $r, s \in \mathbb{C}$ such that ν is central in D_q.

Exercise 2: *Some q-formulas*
In D_{q^2} and in $\mathcal{U}_q(\mathfrak{sl}_2)$ we prefer to use modified q-integers and q-factorials. Define the following rational functions of an indeterminate q

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]! = [n]\,[n-1]\cdots[2]\,[1] \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}.$$

When $q \in \mathbb{C} \setminus \{0\}$ we denote the values of these rational functions at q by adding a subscript q to the above notations. Recall also that $[\![n]\!] = (1-q^n)/(1-q)$ is a rational function of q and $[\![n]\!]_q$ its value at $q \in \mathbb{C} \setminus \{0\}$.

Show the following properties of the (symmetric) q-integers, q-factorials and q-binomials

(a) $[n] = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$ and $[n]_q = q^{1-n}\,[\![n]\!]_{q^2}$

(b) $[m+n] = q^n\,[m] + q^{-m}\,[n] = q^{-n}\,[m] + q^m\,[n]$

(c) $[l]\,[m-n] + [m]\,[n-l] + [n]\,[l-m] = 0$

(d) $[n] = [2]\,[n-1] - [n-2]$.

Exercise 3: *A finite dimensional quotient of $\mathcal{U}_q(\mathfrak{sl}_2)$ when q is a root of unity*
Let $q \in \mathbb{C} \setminus \{0, 1, -1\}$ and consider the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$.

(a) Prove that for all $k \geq 1$ one has

$$\begin{aligned} E\,F^k - F^k\,E &= \frac{[k]_q}{q - q^{-1}}\,F^{k-1}\left(q^{1-k}\,K - q^{k-1}\,K^{-1}\right) \\ F\,E^k - E^k\,F &= \frac{[k]_q}{q - q^{-1}}\left(q^{k-1}\,K^{-1} - q^{1-k}\,K\right)E^{k-1}. \end{aligned}$$

Now suppose that $q \notin \{+1, -1\}$ is a root of unity and denote by e the smallest positive integer such that $q^e \in \{+1, -1\}$.

(b) Show that the elements E^e, K^e, F^e are central in $\mathcal{U}_q(\mathfrak{sl}_2)$.

(c) Let J be two sided ideal in the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ generated by the central elements E^e, F^e and $K^e - 1$. Show that J is a Hopf ideal in the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$. Show that the quotient Hopf algebra $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2) = \mathcal{U}_q(\mathfrak{sl}_2)\,/\,J$ is finite dimensional.

Exercise 4: *A first step of a calculation for diagonalization of α in D_{q^2}-modules*
Let q be a non-zero complex number which is not a root of unity, and let D_{q^2} be the algebra generated by α, α^{-1}, β, $\tilde{\alpha}$, $\tilde{\alpha}^{-1}$, $\tilde{\beta}$ with relations

$$\begin{aligned} \alpha\alpha^{-1} &= 1 = \alpha^{-1}\alpha & \qquad \tilde{\alpha}\tilde{\alpha}^{-1} &= 1 = \tilde{\alpha}^{-1}\tilde{\alpha} \\ \alpha\beta &= q^2\,\beta\alpha & \tilde{\alpha}\tilde{\beta} &= q^2\,\tilde{\beta}\tilde{\alpha} \\ \alpha\tilde{\beta} &= q^{-2}\,\tilde{\beta}\alpha & \tilde{\alpha}\beta &= q^{-2}\,\beta\tilde{\alpha} \\ \alpha\tilde{\alpha} &= \tilde{\alpha}\alpha & \tilde{\beta}\beta - \beta\tilde{\beta} &= \alpha - \tilde{\alpha}. \end{aligned}$$

(a) Let $c \in D_{q^2}$ be a central element (examples are $\kappa = \alpha\tilde{\alpha}$ and the element ν found in *Exercise 1*). Show that for any irreducible D_{q^2}-module V, there is a constant $\lambda \in \mathbb{C}$ such that on V, the element c acts as $\lambda\,\mathrm{id}_V$.

(b) Suppose that V is a finite dimensional D_{q^2}-module, of dimension d. By considering generalized eigenspaces of α (or of $\tilde{\alpha}$), show that the elements β^k and $\tilde{\beta}^k$ must act as zero on V for any $k \geq d$.

(c) Find polynomials $P(\alpha,\tilde{\alpha})$, $Q(\alpha,\tilde{\alpha})$, $R(\alpha,\tilde{\alpha})$ of α and $\tilde{\alpha}$ such that the following equation holds

$$P(\alpha,\tilde{\alpha})\,\beta^2\,\tilde{\beta}^2 + Q(\alpha,\tilde{\alpha})\,\beta\,\tilde{\beta}^2\,\beta + R(\alpha,\tilde{\alpha})\,\tilde{\beta}^2\,\beta^2 \;=\; (q\,\alpha - q^{-1}\,\tilde{\alpha})\,(\alpha - \tilde{\alpha})\,(q^{-1}\,\alpha - q\,\tilde{\alpha}).$$

(d) Suppose that V is a D_{q^2}-module where the central element $\kappa = \alpha\tilde{\alpha}$ acts as $\lambda\,\mathrm{id}_V$ and where $\tilde{\beta}^2$ acts as zero. Show, using the result of (c), that α and $\tilde{\alpha}$ are diagonalizable on V and the eigenvalues of both are among

$$\pm\sqrt{\lambda}\,q^{-1},\ \pm\sqrt{\lambda},\ \pm\sqrt{\lambda}\,q.$$

Conclude in particular that in any two-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module, K is diagonalizable and its possible eigenvalues are ± 1, $\pm q$, $\pm q^{-1}$.