Problem sheet 1

Exercise 1: *Around the Jordan normal form*
The minimal polynomial of a matrix A is the polynomial q_A of smallest positive degree such that $q_A(A) = 0$, with the coefficient of highest order term equal to 1.

(a) Find two matrices $A, B \in \mathbb{C}^{n\times n}$, which have the same minimal polynomial and the same characteristic polynomial, but which are not similar (*Recall:* A and B are similar if $A = P\,B\,P^{-1}$ for some invertible matrix P).

(b) Show that the Jordan normal form of a matrix $A \in \mathbb{C}^{n\times n}$ is unique up to permutation of the Jordan blocks. In other words, if $C_1 = P_1\,A\,P_1^{-1}$ and $C_2 = P_2\,A\,P_2^{-1}$ are both in Jordan normal form, C_1 with blocks $J_{\lambda_1,n_1}, \ldots J_{\lambda_k;n_k}$ and C_2 with blocks $J_{\lambda'_1,n'_1}, \ldots J_{\lambda'_l;n'_l}$, then $k = l$ and there is a permutation $\sigma \in S_k$ suck that $\lambda_j = \lambda'_{\sigma(j)}$ and $n_j = n'_{\sigma(j)}$ for all $j = 1, 2, \ldots, k$.

(c) Show that any two matrices with the same Jordan normal form up to permutation of blocks are conjugates.

Exercise 2: *Dual representation*
Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ be a representation of G in a finite dimensional (complex) vector space V.

(a) Show that any eigenvalue λ of $\rho(g)$, for any $g \in G$, satisfies $\lambda^{|G|} = 1$.

(b) Recall that the dual space of V is $V^* = \{f : V \to \mathbb{C} \text{ linear map}\}$. For $g \in G$ and $f \in V^*$ define $\rho'(g).f \in V^*$ by the formula

$$\langle \rho'(g).f, v\rangle = \langle f, \rho(g^{-1}).v\rangle \qquad \text{for all } v \in V.$$

Show that $\rho' : G \to \mathrm{GL}(V^*)$ is a representation.

(c) Show that $\mathrm{Tr}(\rho'(g))$ is the complex conjugate of $\mathrm{Tr}(\rho(g))$.

Exercise 3: *A two dimensional irreducible representation of S_3*
Find a two-dimensional irreducible representation of the symmetric group S_3.
(*Hint*: Consider the three-cycles, and see what different transpositions would do to the eigenvectors of a three-cycle.)

Exercise 4: *Dihedral group of order 8*
The group D_4 of symmetries of the square is the group with two generators, r and m, and relations $r^4 = e$, $m^2 = e$, $rmrm = e$.

(a) Find the conjugacy classes of D_4. (Element $g_2 \in G$ is conjugate to $g_1 \in G$ if there exists a $h \in G$ such that $g_2 = h\,g_1\,h^{-1}$. Being conjugate is an equivalence relation and conjugacy classes are the equivalence classes of this equivalence relation.)

(b) Find four non-isomorphic one dimensional representations of D_4.

(c) There exists a unique group homomorphism $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ such that

$$\rho(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho(m) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(here, as usual, we identify linear maps $\mathbb{C}^2 \to \mathbb{C}^2$ with their matrices in the standard basis). Check that this two dimensional representation is irreducible.