Final projects

Choose one of the following population models to study:

- **SIRS, constant population size:**

$$\begin{aligned} \text{S} &= \text{Susceptible host} \\ \text{I} &= \text{Infected host} \\ \text{R} &= \text{Recovered host} \end{aligned}$$

$$\begin{aligned} \text{S} + \text{I} &\overset{\alpha}{\to} 2\text{I} && \text{(transmission)} \\ \text{I} &\overset{\beta}{\to} \text{R} && \text{(recovery)} \\ \text{R} &\overset{\gamma}{\to} \text{S} && \text{(loss of immunity)} \end{aligned}$$

  Notes: The interior equilibrium is typically a focus, but it can also be a node. Try comparing the semi-small system approximations in the two cases.

- **SLIS, constant population size:**

$$\begin{aligned} \text{S} &= \text{Susceptible host} \\ \text{L} &= \text{Latent infected host} \\ \text{I} &= \text{Infectious host} \end{aligned}$$

$$\begin{aligned} \text{S} + \text{I} &\overset{\alpha}{\to} \text{L} + \text{I} && \text{(transmission)} \\ \text{L} &\overset{\beta}{\to} \text{I} && \text{(activation)} \\ \text{I} &\overset{\gamma}{\to} \text{S} && \text{(recovery)} \end{aligned}$$

  Notes: The phase portrait and dynamics of this model are somewhat simpler than those of the others presented here. In particular, the interior equilibrium is always a stable node.

- **SI with host birth & death:**

$$\begin{aligned} \text{S} &= \text{Susceptible host} \\ \text{I} &= \text{Infected host} \\ \text{E} &= \text{Empty site} \end{aligned}$$

$$\begin{aligned} \text{S} + \text{I} &\overset{\alpha}{\to} 2\text{I} && \text{(transmission)} \\ \text{I} &\overset{\beta}{\to} \text{E} + \dagger && \text{(disease induced death)} \\ \text{S} + \text{E} &\overset{\gamma}{\to} \text{S} + \text{S} && \text{(host reproduction)} \\ \text{S} &\overset{\delta}{\to} \text{E} + \dagger && \text{(natural death)} \\ \text{I} &\overset{\delta}{\to} \text{E} + \dagger && \text{(natural death)} \end{aligned}$$

  Notes: In the original project handout this model did not include the empty sites. In addition to the general project exercises below, you may wish to investigate what happens to the deterministic and stochastic dynamics of this model in the limit as the total system size $K = E + I + S$ goes to infinity while the basic host reproduction rate $\gamma_0 = \gamma K$ stays constant.

- **Host-parasitoid:**

$$\begin{aligned} \mathrm{X} &= \text{Host} \\ \mathrm{Y} &= \text{Parasitoid} \end{aligned}$$

$$\begin{array}{rcll} \varnothing & \stackrel{\alpha}{\to} & \mathrm{X} & \text{(host influx)} \\ \mathrm{X} & \stackrel{\beta}{\to} & \dagger & \text{(host death)} \\ \mathrm{X}+\mathrm{Y} & \stackrel{\gamma}{\to} & n\mathrm{Y}+\mathrm{Y} & \text{(host capture \& parasitoid reproduction)} \\ \mathrm{Y} & \stackrel{\delta}{\to} & \dagger & \text{(parasitoid death)} \end{array}$$

  Notes: To see more interesting dynamics near the equilibrium, look for parameter values that give large complex eigenvalues of the Jacobian with a small (negative) real part; decreasing $\beta$ should work well.

- **Site occupancy:**

$$\begin{aligned} \mathrm{S} &= \text{Empty site} \\ \mathrm{X} &= \text{Sessile "female" individual in a site} \\ \mathrm{Y} &= \text{Free-floating "male" individual} \end{aligned}$$

$$\begin{array}{rcll} \mathrm{S}+\mathrm{Y} & \stackrel{\alpha}{\to} & \mathrm{X} & \text{(site colonization)} \\ \mathrm{X}+\mathrm{Y} & \stackrel{\beta}{\to} & \mathrm{X}+2\mathrm{Y} & \text{(sexual reproduction)} \\ \mathrm{X} & \stackrel{\gamma}{\to} & \mathrm{S}+\dagger & \text{(death of site owner)} \\ \mathrm{Y} & \stackrel{\delta}{\to} & \dagger & \text{(natural floater death)} \\ 2\mathrm{Y} & \stackrel{\varepsilon}{\to} & \mathrm{Y}+\dagger & \text{(density dependent floater death)} \end{array}$$

  Notes: The density dependent floater death term was not present in the original project handout. You may wish to see what happens when the density dependent death rate $\varepsilon$ tends to zero. (Hint: In the limit the model not well behaved.) In general, this model has somewhat more complex phase portraits than the others, featuring two distinct interior equilibria which appear via a saddle-node bifurcation.

For your chosen model, carry out the following tasks. In all cases, you should assume mass action dynamics. The tasks marked with (*) require numerical solutions.

1. Formulate a (planar) ODE model and perform a phase-plane analysis. If necessary, also do a local stability analysis of the equilibria and check for the possibility of cycles using Dulac's theorem. (Hint: Try using $U(x, y) = 1/xy$ as a Dulac function.)
2. Formulate a stochastic model for small populations.
3. Formulate a PDE and SDE approximation for semi-large systems.
4. Approximate the quasi-stationary distribution by linearizing the above SDE around the deterministic equilibrium.
5. (*) Estimate the probability of invasion (the meaning of this varies with the project) as a function of initial population size for a given system size.
6. (*) Compute a single sample path of the non-linear SDE from (3) using Euler integration and compare the orbit with the approximate quasi-stationary distribution calculated from the linear SDE (4).
7. (*) Compare visual characteristics of the sample path (6) with plots of the auto-covariance and spectral density of the process.