STOCHASTIC POPULATION MODELS

EXERCISES 7-9

7.
Let $\tilde{f}$ and $\tilde{h}$ denote the Fourier transforms of, respectively, f and h, and prove that:

(a) the Fourier transform and its inverse are linear operators,

(b) $\tilde{\tilde{f}}(t) = 2\pi f(-t)$,

(c) $\widetilde{(\frac{d}{dt}f)}(\omega) = i\omega\tilde{f}(\omega)$,

(d) $\frac{d}{d\omega}\tilde{f}(\omega) = -i\widetilde{(tf)}(\omega)$,

(e) $\tilde{f}_\tau(\omega) = e^{-i\omega\tau}\tilde{f}(\omega)$ where $f_\tau(t) := f(t-\tau)$,

(f) $\widetilde{(f*h)}(\omega) = \tilde{f}(\omega)\tilde{h}(\omega)$ where $(f*h)(t) := \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$,

(g) $\widetilde{(fh)}(\omega) = (\tilde{f}*\tilde{h})(\omega)$,

(h) $\int_{-\infty}^{+\infty} f(t)\tilde{h}(t)dt = \int_{-\infty}^{+\infty} \tilde{f}(t)h(t)dt$.

8.
Let $\delta(t)$ denote the Dirac delta distribution, and prove that:

(a) $\tilde{\delta}(\omega) = 1$ for all ω,

(b) $\tilde{1} = 2\pi\delta(t)$ for all t,

(c) $\widetilde{(t^n)}(\omega) = 2(n+1)!\pi\delta(\omega)/(i\omega)^n \quad$ for $n = 0, 1, \ldots,$

(d) $\widetilde{e^{i\omega_0 t}}(\omega) = 2\pi\delta(\omega-\omega_0)$,

(e) $\widetilde{\cos(\omega_0 t)} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$.

9.
Consider the plant population model from exercise 6:

$$\frac{dp}{dt} = \frac{\alpha\beta e^{-\varepsilon\tau}}{\delta}p_\tau(e_0 - p) - \gamma p$$

Assume the populations is near its equilibrium and calculate the transfer function $T(\omega)$ for small fluctuations in

(a) α, the *per capita* rate of seed production,

(b) β, the colonization rate of safe sites,

(c) ε the death rate of dormant seeds,

(d) τ the length of the dormancy period.

Plot the gain $|T(\omega)|$ and the phase-shift $\arg T(\omega)$ as functions of the frequency.