DYNAMICS OF LINEAR OPERATORS, AUTUMN 2010

Solutions to Exercises 6

(1.) Assume $g$ chaotic and

$$f \circ \varphi = \varphi \circ g$$

where $\varphi : Y \to X$ continuous, $\overline{\varphi(Y)} = X$.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\varphi"'] & Y \arrow[d, "\varphi"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

Recall: $f^n \circ \varphi = \varphi \circ g^n \quad \forall n \geq 0$

Earlier (Exercise 1/5): $f$ is transitive (since $g$ is).

Observe: $y \in per(g) \Rightarrow g^n(y) = y$ for some $n$

$$\Rightarrow f^n(\varphi(y)) = \varphi(g^n(y)) = \varphi(y)$$

$$\Rightarrow \varphi(y) \in per(f).$$

Thus $per(f) \supset \varphi(per(g))$, whence

$$\overline{per(f)} \supset \overline{\varphi(per(g))} \supset \overline{\varphi(\overline{per(g)})} = \overline{\varphi(Y)} = X$$

↑ by continuity

$\Rightarrow$ $per(f)$ is dense.

Thm 5.7 $\Rightarrow$ $f$ is chaotic. $\square$

(2.) Known: $T$ is hypercyclic.

Let $e_\lambda(z) = e^{\lambda z} \Rightarrow e_\lambda \in H(\mathbb{C})$ for all $\lambda \in \mathbb{C}$.

We have $Te_\lambda(z) = e^{\lambda(z+1)} = e^\lambda e^{\lambda z} = e^\lambda e_\lambda(z)$

Iterate $\Rightarrow$ $T^n e_\lambda = (e^\lambda)^n e_\lambda = e^{n\lambda} e_\lambda \quad \forall n \geq 1.$

Hence $e_\lambda \in per(T) \iff e^{n\lambda} = 1$ for some $n \geq 1$

$\iff n\lambda = i \cdot 2\pi k$ for some $k \in \mathbb{Z}$, $n \geq 1$

$\iff \lambda \in 2\pi i \mathbb{Q}$.

Since $per(T)$ is a vector space, this gives

$$per(T) \supset span\{e_\lambda \mid \lambda \in 2\pi i \mathbb{Q}\}$$

Lemma 5.12 $\Rightarrow$ this span is dense $\Rightarrow$ $per(T)$ is dense. $\square$

3. $T$ is hypercyclic (by Exercise 3.4) since

$$2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{n+1}{n} = n+1 \xrightarrow[n\to\infty]{} \infty.$$

<u>Claim:</u> $per(T) = \{\bar{0}\}$.

Suppose to the contrary: $\exists\ x = (x_k) \in \ell^1$ s.t. $\boxed{x_{k_0} \neq 0}$ and $T^m x = x$ for some $m \geq 1$. Then

$$(*) \quad \boxed{T^{mp} x = (T^m)^p x = x \quad \forall p \geq 0}$$

Denote $y^{(n)} = (y_k^{(n)})_{k=1}^{\infty} = T^n x$ for $n \geq 0$.

By definition, $y_k^{(1)} = \frac{k+1}{k} x_{k+1} \quad \forall k \geq 1$

Iterating this we get, for $k \geq 1$,

$$y_k^{(n)} = \frac{k+1}{k} \cdot \frac{k+2}{k+1} \cdot \frac{k+3}{k+2} \cdots \frac{k+n}{k+n-1} x_{k+n} = \frac{k+n}{k} x_{k+n}$$

Hence $(*)$ above $\Rightarrow$

$$x_{k_0} = y_{k_0}^{(mp)} = \frac{k_0 + mp}{k_0} x_{k_0+mp} \quad \forall p \geq 0$$

$$\Rightarrow \quad x_{k_0+mp} = \frac{k_0}{k_0+mp} x_{k_0} \quad \forall p \geq 0$$

$$\Rightarrow \quad \|x\|_1 = \sum_{k=1}^{\infty} |x_k| \geq \sum_{p=0}^{\infty} |x_{k_0+mp}|$$

$$= k_0 |x_{k_0}| \sum_{p=0}^{\infty} \frac{1}{k_0+mp} = \infty$$

a contradiction. $\square$

4. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

$\boxed{(ii) \Rightarrow (iii)}$ Banach-Steinhaus theorem applied to the family $\{T^n \mid n \geq 0\}$ (see Functional Analysis course)

$\boxed{(i) \Rightarrow (ii)}$ By sensitive dependence on initial cond., $\exists\, \delta > 0$ and vectors $y_k \in B(\bar{0}, \frac{1}{k})$, $k = 1, 2, \ldots$, s.t.

$$\|T^{n_k} y_k\| = \|T^{n_k} y_k - T^{n_k} \bar{0}\| > \delta \quad \text{for some } n_k\text{'s}$$

Then $\|T^{n_k}\| > k\delta$ $\forall k$, whence (ii) holds.

$\boxed{(iii) \Rightarrow (i)}$ Let $x \in X$, $\varepsilon > 0$ be given.

(iii) $\Rightarrow$ $\exists\, z \in X$ s.t. $\sup_n \|T^n z\| = \infty$

$$\Rightarrow \quad \|T^{n_0} z\| > \frac{2\|z\|}{\varepsilon} \quad \text{for some } n_0$$

Let $z' = \frac{\varepsilon}{2\|z\|} z \Rightarrow \|z'\| = \frac{\varepsilon}{2}$ and

$$\|T^{n_0} z'\| = \frac{\varepsilon}{2\|z\|} \|T^{n_0} z\| > 1$$

Now $y = x + z'$ satisfies $\|y - x\| = \|z'\| < \varepsilon$ and

$$\|T^{n_0} y - T^{n_0} x\| = \|T^{n_0} z'\| > 1.$$

Hence (i) holds (sensitivity constant $\delta = 1$). $\square$

5. $\boxed{\supset}$ If $Tx = \lambda x$ with $\lambda^n = 1$, then $T^n x = \lambda^n x = x$
$\Rightarrow$ $x \in per(T)$.

Also, $per(T)$ is a vector space:

- $T^n x = x$, $\lambda \in \mathbb{C}$ $\Rightarrow$ $T^n(\lambda x) = \lambda x$
- $T^m x = x$, $T^n y = y$ $\Rightarrow$
$T^{mn}(x+y) = (T^m)^n x + (T^n)^m y = x + y$

$\boxed{\subset}$ Let $T^n x = x$, i.e. $(T^n - I)x = \bar{0}$ where $n \geq 1$.

Write $z^n - 1 = (z - \lambda_1) \cdots (z - \lambda_n)$
where $\lambda_1, \dots, \lambda_n$ are the $n^{th}$ roots of unity
($n$ distinct numbers with $\lambda_k^n = 1$ $\forall k$)

Let $p_k(z) = \prod_{j \neq k} (z - \lambda_j)$ for $k = 1, 2, \dots, n$ $(*)$

Lemma below $\Rightarrow$ $\exists$ scalars $a_1, \dots, a_n \in \mathbb{C}$ s.t.
$$\sum_{k=1}^{n} a_k p_k(z) \equiv 1 \quad \text{whence} \quad I = \sum_{k=1}^{n} a_k p_k(T).$$

Therefore $x = \sum_{k=1}^{n} a_k p_k(T) x$.

This belongs to the required span, since
$(T - \lambda_k I) p_k(T) x = (T^n - I) x = \bar{0}$
$\Rightarrow$ $T(p_k(T)x) = \lambda_k p_k(T) x$. $\square$

Lemma. Let $\mathcal{P}_{n-1} = \{ p \mid p \text{ polynomial of degree} \leq n-1 \}$. Then $\{p_1, \dots, p_n\}$ (defined in $(*)$ above) is a basis for $\mathcal{P}_{n-1}$.

Proof. Note: $\dim(\mathcal{P}_{n-1}) = n$ (natural basis: $1, z, z^2, \dots, z^{n-1}$)
Suppose $a_1, \dots, a_n \in \mathbb{C}$ satisfy $\sum_{k=1}^{n} a_k p_k(z) \equiv 0$.
Substitute here $z = \lambda_\ell$ $\Rightarrow$ $p_k(\lambda_\ell) = 0$ $\forall k \neq \ell$, $p_\ell(\lambda_\ell) \neq 0$
$\Rightarrow$ must have $a_\ell = 0$. Thus $a_1 = \dots = a_n = 0$. $\boxed{\text{Lemma}}$