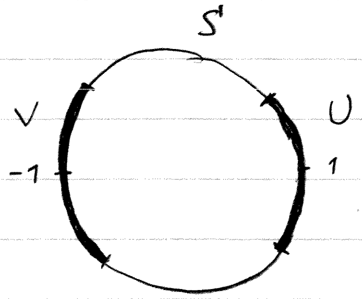
DYNAMICS OF LINEAR OPERATORS, AUTUMN 2010

Solutions to Exercises 5

(1.) **Claim.** f is not weakly mixing



Define
$$U = \{ e^{i\theta} \mid -\tfrac{\pi}{4} < \theta < \tfrac{\pi}{4} \}$$
$$V = \{ e^{i\theta} \mid \tfrac{3\pi}{4} < \theta < \tfrac{5\pi}{4} \}$$

Note: $f^n(e^{i\theta}) = e^{i\cdot(2\pi n\alpha + \theta)}$

In particular $f^n(1) = e^{i\cdot 2\pi n \alpha}$

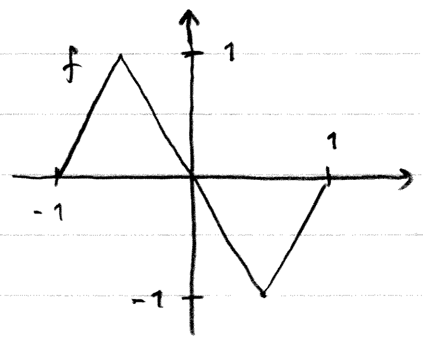
Assume $f^n(U) \cap U \neq \phi$ for some n

$\Rightarrow$ necessarily $|2\pi n\alpha| = |2\pi n\alpha - 0| < \frac{\pi}{2}$ (mod 2π)

But then $|2\pi n\alpha - \pi| > \frac{\pi}{2}$ (mod 2π)

$\Rightarrow$ $f^n(U) \cap V = \phi$. $\square$

(2.) **Recall:** $\begin{cases} f([-1,0]) \subset [0,1] \\ f([0,1]) \subset [-1,0] \end{cases}$



$\Rightarrow$ $\begin{cases} f^n([0,1]) \subset [0,1] & \text{for even } n \\ f^n([0,1]) \subset [-1,0] & \text{for odd } n \end{cases}$

Thus, for every $n \geq 0$, we have

either $f^n(]0,1[) \cap]-1,0[= \phi$ (n even)

or $f^n(]0,1[) \cap]0,1[= \phi$ (n odd)

$\Rightarrow$ f is not weakly mixing. $\square$

(3.) By assumption, $\exists$ dense sets $X_0, Y_0 \subset X$, indices $n_1 < n_2 < \cdots$ and mappings $S_{n_k} : Y_0 \to X$ such that

(i) $\forall x \in X_0 : \quad T^{n_k} x \longrightarrow \bar{0}$

(ii) $\forall y \in Y_0 : \quad S_{n_k} y \longrightarrow \bar{0}$

(iii) $\forall y \in Y_0 : \quad T^{n_k} S_{n_k} y \longrightarrow y$

Then $X_0 \times X_0$, $Y_0 \times Y_0$ are dense in $X \times X$, $S_{n_k} \times S_{n_k} : Y_0 \times Y_0 \longrightarrow X \times X$, and

(i) $\forall (x, x') \in X_0 \times X_0$:
$$(T \times T)^{n_k}(x, x') = (T^{n_k} x, T^{n_k} x') \to (\bar{0}, \bar{0})$$

(ii) $\forall (y, y') \in Y_0 \times Y_0$:
$$(S_{n_k} \times S_{n_k})(y, y') = (S_{n_k} y, S_{n_k} y') \longrightarrow (\bar{0}, \bar{0})$$

(iii) $\forall (y, y') \in Y_0 \times Y_0$:
$$(T \times T)^{n_k}(S_{n_k} \times S_{n_k})(y, y') = (T^{n_k} S_{n_k} y, T^{n_k} S_{n_k} y') \longrightarrow (y, y')$$

Hence $T \times T$ satisfies the Hypercyclicity criterion. $\square$

(4.) a) We repeat the proof (Kitai's criterion) used for D.

Let $X_0 = Y_0$ = analytic polynomials
$S : Y_0 \to Y_0$, $\quad Sp(z) = \lambda^{-1} \int_0^z p$

Then: (i) for all $p \in X_0$, $(\lambda D)^n = \lambda^n D^n p = 0$ for $n > \deg(p)$.

(iii) for all $p \in Y_0$, $(\lambda D) S p = D \int_0^z p = p$

To see that (ii) $S^n p \longrightarrow 0$ in $H(\mathbb{C})$ for all $p \in Y_0$,
we may assume $p(z) = z^k$ for some $k \geq 0$.
Then:
$$S^n p(z) = \lambda^{-n} \frac{z^{k+n}}{(k+1)\cdots(k+n)}$$

Hence for $|z| \leq R$,

$$|S^n p(z)| \leq \lambda^{-n} \frac{R^{k+n}}{(k+1)\cdots(k+n)} \leq R^k \frac{(R/\lambda)^n}{n!} \xrightarrow[n\to\infty]{} 0$$

since $\sum_{n=1}^{\infty} \frac{(R/\lambda)^n}{n!} < \infty$ (by ratio test, for example).

Thus $S^n p \longrightarrow 0$ in $H(\mathbb{C})$.

In conclusion, λD satisfies Kitai's criterion, whence it is mixing. $\square$

b) We assume $X \neq \{\vec{0}\}$ and $T \in L(X)$.

If $T \equiv 0$, clearly it is not hypercyclic.
Assume $T \not\equiv 0$, i.e. $\|T\| > 0$. Choose $0 < \lambda \leq 1/\|T\|$.
Then $\|\lambda T\| = |\lambda| \, \|T\| \leq 1 \Rightarrow \lambda T$ contractive.
Exercise 2/1 $\Rightarrow$ λT not hypercyclic. $\square$

5. Suppose $x_0 \in X$ isolated, i.e. $\{x_0\} \subset X$ open.

By Exercise 1/3, $f^n\{x_0\} \cap \{x_0\} \neq \phi$ for infinitely many n
$\Rightarrow \exists\, n_0 \geq 1$ s.t. $f^{n_0}(x_0) = x_0$.

$$\left.\begin{array}{l} \text{Then } \operatorname{orb}(f, x_0) = \{x_0, f(x_0), \ldots, f^{n_0-1}(x_0)\} \text{ is } \underline{\text{finite}}. \\ \text{Prop. 1.2.} \Rightarrow \operatorname{orb}(f, x_0) = \bigcup_{n=0}^{\infty} f^n\{x_0\} \ \underline{\text{dense}} \text{ in } X. \end{array}\right\} \Rightarrow$$

must have $X = orb(f, x_0)$; in particular, X is finite.

It also follows from above that f is surjective

$\Rightarrow$ f bijective (because X finite). $\square$

Remark. Since X is finite, every $x \in X$ is isolated

$\Rightarrow$ $orb(f, x) = X$ for every $x \in X$ (as above).

6. $\boxed{(i) \Rightarrow (ii) \Rightarrow (iii)}$ clear!

$\boxed{(iii) \Rightarrow (ii)}$ Let U, V_1, V_2 be given.

(iii) $\Rightarrow$ $\exists\, k \geq 0$ s.t. $W = U \cap f^{-k}(V_1) \neq \emptyset$

Fact given (see e.g. solution to Exercise 1/3) $\Rightarrow$ $V = f^{-k}(V_2) \neq \emptyset$

(iii) $\Rightarrow$ $\exists\, n \geq 0$ s.t.

$$\begin{cases} f^n(W) \cap W \neq \emptyset \\ f^n(W) \cap V \neq \emptyset \end{cases} \iff \begin{cases} W \cap f^{-n}(W) \neq \emptyset \\ W \cap f^{-n}(V) \neq \emptyset \end{cases} \qquad (*)$$

On the other hand,

$$\begin{cases} W \cap f^{-n}(W) \subset U \cap f^{-n}(f^{-k}(V_1)) = U \cap f^{-(n+k)}(V_1) \\ W \cap f^{-n}(V) \subset U \cap f^{-n}(f^{-k}(V_2)) = U \cap f^{-(n+k)}(V_2) \end{cases}$$

Hence $(*)$ $\Rightarrow$ $\begin{cases} U \cap f^{-(n+k)}(V_1) \neq \emptyset \\ U \cap f^{-(n+k)}(V_2) \neq \emptyset \end{cases} \Rightarrow \begin{cases} f^{n+k}(U) \cap V_1 \neq \emptyset \\ f^{n+k}(U) \cap V_2 \neq \emptyset \end{cases}$

$\boxed{(ii) \Rightarrow (i)}$ Let U_1, V_1, U_2, V_2 be given.

(ii) $\Rightarrow$ $\exists\, k \geq 0$ s.t. $U = U_1 \cap f^{-k}(U_2) \neq \emptyset$

Fact $\Rightarrow$ $f^{-k}(V_2) \neq \emptyset$

(ii) again $\Rightarrow$ $\exists\, n \geq 0$ s.t. $\begin{cases} f^n(U) \cap V_1 \neq \emptyset & (1) \\ f^n(U) \cap f^{-k}(V_2) \neq \emptyset & (2) \end{cases}$

Then we get

$$f^n(U_1) \cap V_1 \supset f^n(U) \cap V_1 \neq \emptyset \quad \text{by } (1)$$

and

$$\begin{aligned} f^n(U_2) \cap V_2 &\supset f^n\left(f^k(f^{-k}(U_2))\right) \cap V_2 \\ &\supset f^n\left(f^k(U)\right) \cap V_2 \\ &= f^k\left(f^n(U)\right) \cap V_2 \neq \emptyset \quad \text{by } (2) \end{aligned}$$

$\square$