DYNAMICS OF LINEAR OPERATORS, AUTUMN 2010

Solutions to Exercises 4

(1.) If $\dim(X) < \infty$ (but $X \neq \{\bar{0}\}$), then Corollary 2.4 $\Rightarrow$ $T \in L(X)$ is not hypercyclic. (Corollary is valid for Fréchet spaces as well!)

Suppose $\dim(X) = \infty$. Take $x \in X$. Then

$$orb(T, x) = \{x, Tx, T^2x, \dots\} \subset span(\{x\} \cup T(X)) \equiv M$$

where $\dim(M) \leq 1 + \dim(T(X)) < \infty$, and so $M \neq X$.

Fact (functional analysis course!): finite-dimensional subspace is always closed. Thus $\overline{orb(T,x)} \subset M \neq X$. $\square$

(2.) Observe: $(\lambda T)^p = \lambda^p T^p = T^p$. Apply Ansari's thm twice to get $HC(\lambda T) = HC((\lambda T)^p) = HC(T^p) = HC(T)$. $\square$

Remark. It can be shown that $HC(\lambda T) = HC(T)$ for every $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. The above covers only the cases $\lambda = e^{i \cdot 2\pi\alpha}$ with $\alpha \in \mathbb{Q}$.

(3.) Let $A = \{p(T)x \mid p \neq 0 \text{ polynomial}\}$, where $x \in HC(T)$.

Exercise 3.5 $\Rightarrow$ $A \subset HC(T)$.

Lemma 3.2 $\Rightarrow$ A is connected and $\bar{A} = X$.

Thus $A \subset HC(T) \subset \bar{A}$.

Topology fact [Väisälä, Topo I, 14.11] $\Rightarrow$ since A is connected, $HC(T)$ is connected. $\square$

(4.) We assume $\bigcup_{j=1}^{\infty} F_j = X$, where $F_j = \overline{\mathrm{orb}(T, x_j)}$.

We have $\bigcap_{j=1}^{\infty} (X \setminus F_j) = X \setminus \bigcup_{j=1}^{\infty} F_j = \emptyset$;

in particular, this is <u>not dense</u>. Hence Baire's thm

$\Rightarrow$ $\exists j$ s.t. the open set $X \setminus F_j$ is not dense

$\Rightarrow$ $X \neq \overline{X \setminus F_j} = X \setminus \mathrm{int}(F_j)$

$\Rightarrow$ $\mathrm{int}(F_j) \neq \emptyset$.

Bourdon - Feldman thm (Thm 3.3) $\Rightarrow$ $x_j \in HC(T)$. □

(5.) (ii) $\Rightarrow$ (i): Observe: $\lambda T^n(U) \cap V \neq \emptyset \Leftrightarrow U \cap (\lambda T^n)^{-1}(V) \neq \emptyset$

Thus (ii) $\Rightarrow$

$$(*) \left\{ \begin{array}{l} \bigcup_{n=0}^{\infty} \bigcup_{\lambda \in \mathbb{K}} (\lambda T^n)^{-1}(V) \text{ dense} \\ \text{for every nonempty open } V \subset X. \end{array} \right.$$

Let S be the set of those $x \in X$ for which $\{\lambda T^n x \mid \lambda \in \mathbb{K}, n \geq 0\}$ is dense. Let $(V_k)_{k=1}^{\infty}$ be a basis for the topology of X. Then:

$$\begin{aligned} x \in S &\Leftrightarrow \forall k \geq 1 \ \exists \lambda \in \mathbb{K}, n \geq 0 \text{ s.t. } \\ &\qquad \lambda T^n x \in V_k, \text{ i.e. } x \in (\lambda T^n)^{-1}(V_k) \\ &\Leftrightarrow \forall k \geq 1 \quad x \in \boxed{\bigcup_{n=0}^{\infty} \bigcup_{\lambda \in \mathbb{K}} (\lambda T^n)^{-1}(V_k)} \\ &\Leftrightarrow x \in \bigcap_{k=1}^{\infty} \boxed{} \end{aligned}$$

That is,

$$S = \bigcap_{k=1}^{\infty} \underbrace{\bigcup_{n=0}^{\infty} \bigcup_{\lambda \in \mathbb{K}} (\lambda T^n)^{-1}(V_k)}_{\text{dense by } (*)}$$

Baire's thm $\Rightarrow$ S is dense; in particular $S \neq \emptyset$.

(i) $\Rightarrow$ (ii): Assume $A = \{ \lambda T^n x \mid \lambda \in \mathbb{K},\ n \geq 0 \}$ is dense.

Let $U, V \subset X$ be open and nonempty.

By density, $\exists\ \lambda_0, n_0$ s.t. $\lambda_0 T^{n_0} x \in U$, $\lambda_0 T^{n_0} x \neq \bar{0}$.

Consider the set

$$\begin{aligned} B &= \{ \lambda T^n (\lambda_0 T^{n_0} x) \mid \lambda \in \mathbb{K},\ n \geq 0 \} \\ &= \{ \lambda T^{n_0 + n} x \mid \lambda \in \mathbb{K},\ n \geq 0 \} \qquad \text{(note that } \lambda_0 \neq 0\text{)} \end{aligned}$$

Claim: B is dense in X.

Pf of claim: If $\dim(X) = 1$, this is clear (just use $n = 0$).

Assume $\dim(X) > 1$. We have $B \supset A \setminus E$, where

$$\begin{aligned} E &= \{ \lambda T^n x \mid \lambda \in \mathbb{K},\ n = 0, 1, \dots, n_0 - 1 \} \\ &= \bigcup_{n=0}^{n_0 - 1} \operatorname{span}(T^n x). \end{aligned}$$

Here $\operatorname{span}(T^n x)$ is closed (since finite-dimensional) and has empty interior in X (since proper subspace) for every n

$\Rightarrow$ E is closed, $\operatorname{int} E = \emptyset$.

Thus if $W \subset X$ is $\neq \emptyset$ and open $\Rightarrow$ $W \setminus E$ is $\neq \emptyset$ and open

$\Rightarrow$ $\emptyset \neq A \cap (W \setminus E) = (A \setminus E) \cap W \subset B \cap W$.

A dense

$\boxed{\text{claim}}$

Since B is dense, $\exists\ \lambda \in \mathbb{K}$, $n \geq 0$ s.t. $\lambda T^n (\lambda_0 T^{n_0} x) \in V$.

Here $\lambda_0 T^{n_0} x \in U$ $\Rightarrow$ $\lambda T^n (U) \cap V \neq \emptyset$. $\square$

6. $\boxed{\text{in } \mathbb{R}}$ Define $Tx = ax$ $\forall x \in \mathbb{R}$ with $a \in \mathbb{R} \setminus \{0\}$

$\Rightarrow$ $\{\lambda T(1) \mid \lambda \in \mathbb{R}\} = \{\lambda a \mid \lambda \in \mathbb{R}\} = \mathbb{R}$

$\Rightarrow$ T supercyclic.

$\boxed{\text{in } \mathbb{R}^2}$ Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by (using complex notation)

$$Tz = e^{i \cdot 2\pi\alpha} z \quad \text{for } z = x + iy = (x, y) \in \mathbb{R}^2$$

i.e. $$T(x,y) = \begin{bmatrix} \cos 2\pi\alpha & -\sin 2\pi\alpha \\ \sin 2\pi\alpha & \cos 2\pi\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (rotation through the angle $2\pi\alpha$).

<u>Claim:</u> $A = \{\lambda T^n(1) \mid \lambda \in \mathbb{R},\ n \geq 0\}$ is dense ($\Rightarrow$ T supercyclic)

<u>Proof:</u> Here $\lambda T^n(1) = \lambda e^{i \cdot 2\pi n\alpha} = (\lambda \cos 2\pi n\alpha, \lambda \sin 2\pi n\alpha)$.

Let $z = re^{i\theta} = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$ be given.

We know (Exercise 1/2): $\exists$ integers $n_1, n_2, \ldots$ s.t.

$T^{n_k}(1) = e^{i \cdot 2\pi n_k \alpha} \longrightarrow e^{i\theta}$ as $k \to \infty$.

Then $\underbrace{r\, T^{n_k}(1)}_{\in A} \longrightarrow re^{i\theta} = z$ as $k \to \infty$

$\Rightarrow$ $z \in \overline{A}$. $\therefore$ $\overline{A} = \mathbb{R}^2$. $\square$