DYNAMICS OF LINEAR OPERATORS, AUTUMN 2010

Solutions to Exercises 3

1. Enough to show that T is transitive (by Birkhoff's thm).

Let nonempty open sets $U, V \subset X$ be given.

By density, $\exists\, x \in X_0 \cap U$, $y \in Y_0 \cap V$

Choose (u_n) s.t. $u_n \to \bar{0}$, $T^n u_n \to y$.

Then:
$$\begin{cases} x + u_n \longrightarrow x + \bar{0} = x \\ T^n(x+u_n) = T^n x + T^n u_n \to \bar{0} + y = y \end{cases}$$

$$\Rightarrow \begin{cases} x + u_n \in U \\ T^n(x+u_n) \in V \end{cases} \quad \text{for large enough } n$$

$$\Rightarrow \quad T^n(U) \cap V \neq \emptyset \quad \text{for large enough } n$$

$\square$

2. Choose $X_0 = Y_0 = \{ f_{p,\alpha,k} \mid p \text{ polynomial}, \alpha > 0, k \geq 0 \}$

where $f_{p,\alpha,k}(z) = p(z)\, e^{-\alpha(z-k)^2}$

Claim 1. X_0 is dense in $H(\mathbb{C})$.

Let p be a polynomial. We show: $f_{p,\alpha,0} \to p$ in $H(\mathbb{C})$ as $\alpha \to 0+$.

Let $R > 0$ and $|z| \leq R$. Then

$$|f_{p,\alpha,0}(z) - p(z)| = |p(z)|\,|e^{-\alpha z^2} - 1|$$
$$= |p(z)| \cdot \left| \sum_{j=1}^{\infty} \frac{(-\alpha z^2)^j}{j!} \right| \leq P_R \sum_{j=1}^{\infty} \frac{\alpha^j R^{2j}}{j!}$$
$$= P_R\left(e^{\alpha R^2} - 1\right) \longrightarrow 0, \text{ as } \alpha \to 0+$$

where $P_R = \sup_{|z| \leq R} |p(z)| < \infty$.

$\therefore$ $f_{p,\alpha,0} \to p$ uniformly in every compact set, i.e. in $H(\mathbb{C})$.

Since polynomials are dense in $H(\mathbb{C})$ (see the proof of Prop. 2.11), we conclude that X_0 is dense.

Claim 2. $T: H(\mathbb{C}) \to H(\mathbb{C})$, $Tf(z) = f(z+1)$, is hypercyclic.

Define $S: H(\mathbb{C}) \to H(\mathbb{C})$, $Sf(z) = f(z-1)$.

Clearly $S(X_0) \subset X_0$ and $TSf = f \quad \forall f \in X_0$ (even $H(\mathbb{C})$)

We want to show: $T^n f_{p,\alpha,k} \xrightarrow[n\to\infty]{} 0$, $S^n f_{p,\alpha,k} \xrightarrow[n\to\infty]{} 0$ in $H(\mathbb{C})$, that is,

$$(*) \qquad f_{p,\alpha,k}(z \pm n) \xrightarrow[n\to\infty]{} 0 \quad \text{unif. on compacts}$$

for every $f_{p,\alpha,k} \in X_0$. Then Kitai's criterion says that T is hypercyclic.

Proof of $(*)$: Fix a polynomial p (of degree M) and $\alpha > 0$, $k \geq 0$.

two facts:
- $|p(w)| \leq C|w|^M$ for $|w| \geq 1$ ($C > 0$ const.)
- $|e^w| = e^{\operatorname{Re} w}$ for $w \in \mathbb{C}$

Hence if $R > 1$ is given, and $|z| \leq R$, $n \geq 2R + 2k$, we may estimate (writing $z = x + iy$):

$$\begin{aligned}
|f_{p,\alpha,k}(z\pm n)| &= |p(z\pm n)|\, e^{-\alpha \operatorname{Re}(z-k\pm n)^2} \\
&= |p(z\pm n)|\, e^{-\alpha[(x-k\pm n)^2 - y^2]} \\
&= |p(z\pm n)|\, e^{\alpha y^2} e^{-\alpha[n \pm (x-k)]^2} \\
&\leq C\,|z\pm n|^M e^{\alpha R^2} e^{-\alpha n/2} \\
&\leq C e^{\alpha R^2} \frac{(R+n)^M}{e^{\alpha n/2}} \xrightarrow[n\to\infty]{} 0 \quad \text{(by routine calculus!)}
\end{aligned}$$

$$\boxed{\begin{aligned} &|n \pm (x-k)| \\ &\geq n - R - k \\ &\geq n - \tfrac{1}{2}n = \tfrac{1}{2}n \end{aligned}}$$

Therefore $f_{p,\alpha,k}(z \pm n) \xrightarrow[n\to\infty]{} 0$ uniformly for $|z| \leq R$. $\square$

(3.) **Recall fact:** For a linear mapping $T : \omega \to \omega$ we have:

T is continuous $\iff$ it is continuous at $\bar{0}$ $\iff$

$Tx^{(n)} \to \bar{0}$ whenever $x^{(n)} \to \bar{0}$ in ω.

Also, $x^{(n)} \to \bar{0}$ in ω means $x_k^{(n)} \xrightarrow[n\to\infty]{} 0 \quad \forall k$

where we have written $x^{(n)} = (x_k^{(n)})_{k=1}^{\infty}$.

a) Assume $x^{(n)} \longrightarrow \bar{0}$ in ω, $x^{(n)} = (x_k^{(n)})_{k=1}^{\infty}$

Let $y^{(n)} = (y_k^{(n)})_{k=1}^{\infty} = B_a x^{(n)}$

Then $y_k^{(n)} = a_{k+1} x_{k+1}^{(n)} \xrightarrow[n\to\infty]{} 0 \quad \forall k$

$\Rightarrow \quad y^{(n)} \longrightarrow \bar{0}$ in ω, $\quad \therefore \; B_a : \omega \to \omega$ is cont.

b) **Known:** $c_{00} = \{(x_1, \dots, x_m, 0, 0, \dots) \mid x_1, \dots, x_m \in \mathbb{K} ; \; m \geq 1\}$

is dense in ω (see Exercise 2.6).

We apply Kitai's criterion with

$$S(y_1, y_2, y_3, \dots) = \left(0, \frac{y_1}{a_2}, \frac{y_2}{a_3}, \frac{y_3}{a_4}, \dots\right)$$

Obviously $B_a S y = y \quad \forall y \in c_{00}$ (or even $y \in \omega$).

If $x = (x_1, x_2, \dots, x_m, 0, 0, \dots) \in c_{00}$ given, clearly

$$B_a^n x = \bar{0} \quad \forall n \geq m, \quad \text{whence} \quad B_a^n x \xrightarrow[n\to\infty]{} \bar{0}$$

Also, if $y \in c_{00}$ (or even $y \in \omega$), and

$y^{(n)} = (y_k^{(n)})_{k=1}^{\infty} = S^n y$, then

$$y_k^{(n)} = 0 \quad \forall \, k \leq n \quad \Rightarrow \quad y_k^{(n)} \xrightarrow[n\to\infty]{} 0 \quad \forall k \geq 1$$

$$\Rightarrow \quad y^{(n)} = S^n y \xrightarrow[n\to\infty]{} \bar{0}$$

Hence Kitai's conditions are satisfied! $\square$

(4.) Assume: (i) $0 < |a_k| \le M$ (where $M > 1$) $\forall k \ge 1$

(ii) $\sup_{n \ge 1} |a_1 a_2 \cdots a_n| = \infty$

Lemma. $\exists$ integers $0 < n_1 < n_2 < \cdots$ s.t.

$$\lim_{k\to\infty} \underbrace{|a_{j+1} a_{j+2} \cdots a_{j+n_k}|}_{\substack{n_k \text{ consecutive} \\ \text{factors !}}} = \infty \quad \forall j \ge 0$$

Pf of Lemma. Assumption (ii) $\Rightarrow$ $\exists$ integers $m_k \to \infty$ s.t.

$$(*) \quad |a_1 a_2 \cdots a_{m_k}| \ge M^{2k} \quad \forall k \ge 1.$$

We may assume that $m_1 \ge 2$ and $m_{k+1} > m_k + 1$ $\forall k \ge 1$ (pass to a subsequence if necessary).

Let $n_k = m_k - k$ for $k \ge 1$. Then $n_1 \ge 1$ and

$$n_{k+1} - n_k = m_{k+1} - m_k - 1 > 0 \quad \forall k \ge 1$$

$\Rightarrow$ (n_k) is strictly increasing and $n_k \to \infty$ as $k \to \infty$.

Fix $j \ge 0$. For $k > j$ we have $j + n_k < k + n_k = m_k$ and so we may write

$$|a_{j+1} a_{j+2} \cdots a_{j+n_k}| = \frac{|a_1 a_2 \cdots a_{m_k}|}{|a_1 \cdots a_j|\,|a_{j+n_k+1} \cdots a_{m_k}|}$$

Here numerator is $\ge M^{2k}$ by $(*)$, and denominator has $j + (m_k - j - n_k) = k$ factors, each of which has modulus $\le M$ by (i). Hence

$$|a_{j+1} a_{j+2} \cdots a_{j+n_k}| \ge \frac{M^{2k}}{M^k} = M^k \quad \forall k > j$$

where $M^k \to \infty$ as $k \to \infty$. $\boxed{\text{Lemma}}$

<u>Claim.</u> $B_a : \ell^p \to \ell^p$ $(1 \le p < \infty)$, $(x_1, x_2, \dots) \mapsto (a_2 x_2, a_3 x_3, \dots)$

is hypercyclic (still assuming (i) and (ii)).

<u>Proof.</u> Define (as in the prev. exercise) $S : c_{00} \to c_{00}$,

(*) $$(y_1, y_2, y_3, \dots) \mapsto \left(0, \frac{y_1}{a_2}, \frac{y_2}{a_3}, \frac{y_3}{a_4}, \dots\right)$$

Recall: c_{00} dense in ℓ^p. Also (as in the prev. exercise)

- $\forall x \in c_{00}$: $B_a^n x = \bar{0}$ for large n, so $B_a^n x \xrightarrow[n\to\infty]{} \bar{0}$ in ℓ^p
- $B_a S y = y$ $\forall y \in c_{00}$, and hence $B_a^n S^n y = y$ $\forall y \in c_{00}$

We show: $S^{n_k} y \xrightarrow[k\to\infty]{} \bar{0}$ $\forall y \in c_{00}$, where (n_k) is given by the above lemma. Then Hypercyclicity Criterion $\Rightarrow$ B_a hypercyclic.

Enough to check: $S^{n_k} e_j \longrightarrow \bar{0}$ in ℓ^p for every standard unit vector

$$e_j = (0, \dots, 0, \overset{j\text{:th}}{1}, 0, 0, \dots), \quad j \ge 1$$

because each $y \in c_{00}$ is a finite linear combination of such e_j's.

Observe by (*): $$S e_j = \frac{1}{a_{j+1}} e_{j+1}$$

and by iterating,

$$S^n e_j = \frac{1}{a_{j+1} a_{j+2} \cdots a_{j+n}} e_{j+n}$$

In particular,

$$\| S^{n_k} e_j \|_p = \frac{1}{|a_{j+1} a_{j+2} \cdots a_{j+n_k}|} \overbrace{\| e_{j+n_k} \|_p}^{=1}$$

$$\xrightarrow[k\to\infty]{} \infty \quad \text{by Lemma.}$$

$\boxed{\text{Claim}}$

<u>Remark</u>. Also the converse implication holds: if B_a is hypercyclic on ℓ^p $(1 \le p < \infty)$, then (ii) holds.

(5.) Let $y = p(T)x \in M \setminus \{\bar{0}\}$ be arbitrary. Necessarily $p \not\equiv 0$.

We have

$$T^n y = T^n p(T) x \overset{(*)}{=} p(T)(T^n x) \qquad \forall n \geq 0$$

$$\Rightarrow \quad orb(T,y) = p(T)(orb(T,x))$$

Since $p(T)$ is continuous and $\overline{p(T)(X)} = X$ by Lemma 3.1, we get

$$\overline{orb(T,y)} = \overline{p(T)\left(\overline{orb(T,x)}\right)} \underset{x \in HC(T)}{=} \overline{p(T)(X)} = X.$$

Thus $y \in HC(T)$. $\square$

(*) <u>Observe</u>: Writing $p(T) = \sum_{j=0}^{k} a_j T^j$ for some $a_0, \dots, a_k \in \mathbb{C}$ we have

$$T^n p(T) = \sum_{j=0}^{k} a_j T^{n+j} = p(T) T^n.$$