DYNAMICS OF LINEAR OPERATORS, AUTUMN 2010

Solutions to Exercises 2

(1.) We are assuming that $X \neq \{\bar{0}\}$.

Take any $x \in X \setminus \{\bar{0}\}$. Then

$$\|T^n x\| \leq \|T\|^n \|x\| \leq \|x\| \quad \forall n \geq 0$$

$\Rightarrow$ $orb(T,x) \subset \overline{B(\bar{0}, \|x\|)} = \{y \in X \mid \|y\| \leq \|x\|\}$

$\Rightarrow$ $\overline{orb(T,x)} \subset \overline{B(\bar{0}, \|x\|)}$

Hence $orb(T,x)$ is not dense in X; for example $2x \notin \overline{orb(T,x)}$. $\square$

(2.) Recall: ℓ^∞ equipped with the norm $\|x\|_\infty = \sup_{k \geq 1} |x_k|$ for $x = (x_k)_{k=1}^\infty \in \ell^\infty$

Let $A = \{(x_k) \in \ell^\infty : |x_k| > 1 \quad \forall k \geq 1\}$. Then:

1°) A is invariant for T, i.e. $T(A) \subset A$:

If $x = (x_k) \in A$ and $y = (y_k) = Tx$, then

$$|y_k| = |2x_{k+1}| = 2|x_{k+1}| > 2 \cdot 1 > 1 \quad \forall k \geq 1$$

$\Rightarrow$ $y \in A$

2°) $\mathrm{int}(A) \neq \emptyset$. In fact, if $z = (2,2,2,\dots) \in \ell^\infty$, then $B(z,1) = \{x \in \ell^\infty : \|x - z\|_\infty < 1\} \subset A$

3°) $\mathrm{int}(\ell^\infty \setminus A) \neq \emptyset$. In fact,

$B(\bar{0}, 1) = \{x \in \ell^\infty : \|x\|_\infty < 1\} \subset \ell^\infty \setminus A$

Hence Prop. 1.2 (iii) fails for T $\Rightarrow$ T <u>non-transitive</u>.

3. We assume that $\exists\, \alpha \in \mathbb{K}$ and $\varphi \in X^* \setminus \{\bar{0}\}$ (a nonzero continuous linear functional $X \to \mathbb{K}$) s.t. $T^* \varphi = \alpha \varphi$.

By iterating, we get $(T^*)^2 \varphi = T^* T^* \varphi = T^*(\alpha\varphi) = \alpha T^* \varphi = \alpha^2 \varphi$, and in general,

$$(*) \qquad \boxed{(T^*)^n \varphi = \alpha^n \varphi} \qquad \text{for } n \geq 0.$$

For each $x \in X$, this gives

$$\begin{aligned} \varphi(T^n x) &= \langle \varphi, T^n x \rangle = \langle (T^n)^* \varphi, x \rangle \\ &= \langle (T^*)^n \varphi, x \rangle \underset{(*)}{=} \langle \alpha^n \varphi, x \rangle \\ &= \alpha^n \varphi(x) \end{aligned}$$

Therefore $\varphi(\mathrm{orb}(x)) = \{ \alpha^n \varphi(x) \mid n \geq 0 \}$.

Observe: $\varphi \neq \bar{0} \Rightarrow \varphi(X) = \mathbb{K}$

Hence, if $x \in HC(T)$, i.e. $\overline{\mathrm{orb}(x)} = X$, we would get

continuity!

$$\begin{aligned} \mathbb{K} = \varphi(X) &= \varphi\left(\overline{\mathrm{orb}(x)}\right) \subset \overline{\varphi(\mathrm{orb}(x))} \\ &= \overline{\{ \alpha^n \varphi(x) \mid n \geq 0 \}} \end{aligned}$$

But this impossible since

- $|\alpha| < 1 \Rightarrow \alpha^n \longrightarrow 0$
- $|\alpha| = 1 \Rightarrow |\alpha^n| = 1 \quad \forall n$
- $|\alpha| > 1 \Rightarrow |\alpha^n| \to \infty$

and so $\{ \alpha^n \varphi(x) \mid n \geq 0 \}$ cannot be dense in $\mathbb{K}$! $\square$

(4.) **Notation:** $A + B = \{x + y \mid x \in A,\ y \in B\}$ for $A, B \subset X$

$z - A = \{z - x \mid x \in A\}$ for $z \in X$, $x \in A$

$z \in X$ given

Claim: $z = x + y$ for some $x \in HC(T)$, $y \in HC(T)$

$\Leftrightarrow$ $x = z - y \in HC(T)$ for some $y \in HC(T)$

Thus it is enough to prove that

$$\boxed{HC(T) \cap (z - HC(T)) \neq \emptyset}$$

Recall: By Birkhoff's thm, $HC(T)$ is a dense G_δ-set:

$$HC(T) = \bigcap_{j=1}^{\infty} U_j \quad \text{where } U_j\text{'s are open \& dense}$$

Then

$$z - HC(T) = \bigcap_{j=1}^{\infty} (z - U_j) = \bigcap_{j=1}^{\infty} V_j$$

where V_j is the image of U_j under the mapping $\phi : X \to X$, $\phi(x) = z - x$.

Since ϕ is a homeomorphism (note that $\phi^{-1} = \phi$), V_j's are open and dense as well. Thus

$$HC(T) \cap (z - HC(T)) = \left(\bigcap_{j=1}^{\infty} U_j\right) \cap \left(\bigcap_{j=1}^{\infty} V_j\right)$$

is the intersection of countably many open and dense sets. Baire's thm $\Rightarrow$ the intersection is a dense G_δ and hence non-empty. $\square$

5. a) • symmetry $d(x,y) = d(y,x)$ follows from

$$p_n(x-y) = p_n(-1\cdot(y-x)) = |-1|\, p_n(y-x) = p_n(y-x)$$

• triangle ineq $d(x,z) \le d(x,y) + d(y,z)$ follows from

$$p_n(x-z) = p_n((x-y)+(y-z)) \le p_n(x-y) + p_n(y-z)$$

• pos. definiteness: $d(x,y) = 0 \iff p_n(x-y) = 0 \ \forall n$

$\overset{*}{\iff} x - y = \bar{0} \iff x = y$ (*since (p_n) separating!)

• transl. invariance obvious

b) "$\Rightarrow$" Assume $d(x_k, y) \to 0$. Fix $n \geq 1$.

$\exists\, k_0 \geq 1$ s.t. $d(x_k, y) < 2^{-n}$ $\forall k > k_0$. Then

$$2^{-n} > d(x_k,y) \geq 2^{-n} \min\{1, p_n(x_k - y)\} \quad \forall k > k_0$$

$\Rightarrow$

$$p_n(x_k - y) \le 2^n d(x_k, y) \quad \forall k > k_0$$

Since $d(x_k,y) \to 0$, must have $p_n(x_k - y) \to 0$ as $k \to \infty$.

"$\Leftarrow$" Assume $p_n(x_k - y) \xrightarrow[k\to\infty]{} 0$ $\forall n \geq 1$. Let $\varepsilon > 0$.

Choose $m \geq 1$ large enough s.t. $\sum_{n=m+1}^{\infty} 2^{-n} = 2^{-m} < \frac{\varepsilon}{2}$

Since $p_n(x_k - y) \to 0$ $\forall n$, $\exists\, k_0 \geq 1$ s.t.

$$p_n(x_k - y) < \frac{\varepsilon}{2} \quad \forall k > k_0, \ n = 1, 2, \dots, m$$

Then for $k > k_0$ we have

$$\begin{aligned} d(x_k, y) &= \sum_{n=1}^{\infty} 2^{-n} \min\{1, p_n(x_k - y)\} \\ &\le \sum_{n=1}^{m} 2^{-n} \cdot \frac{\varepsilon}{2} + \sum_{n=m+1}^{\infty} 2^{-n} \\ &< \frac{\varepsilon}{2} \sum_{n=1}^{\infty} 2^{-n} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$\therefore$ $d(x_k, y) \to 0$. $\square$

(6.) $\omega = \mathbb{K}^{\mathbb{N}} = \{ (x_k) \mid x_k \in \mathbb{K} \quad \forall k \geq 1 \}$

$p_n(x) = \sup\{|x_1|, \dots, |x_n|\}$, $x = (x_k) \in \omega$

Obvious: $(p_n)_{n=1}^{\infty}$ increasing and separating family of seminorms

Metric d defined as

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, p_n(x-y)\}$$

Claim 1. If $x^{(k)} = (x_j^{(k)})_{j=1}^{\infty} \in \omega \quad \forall k \geq 1$

and $y = (y_j)_{j=1}^{\infty} \in \omega$, then

$$d(x^{(k)}, y) \xrightarrow[k\to\infty]{} 0 \iff x_j^{(k)} \xrightarrow[k\to\infty]{} y_j \quad \forall j \geq 1$$

Proof. Exercise 5: $d(x^{(k)}, y) \to 0 \iff$

$$p_n(x^{(k)} - y) = \max\{|x_1^{(k)} - y_1|, \dots, |x_n^{(k)} - y_n|\} \xrightarrow[k\to\infty]{} 0$$

$\forall n \geq 1$

Easy to check that this is equivalent to $|x_j^{(k)} - y_j| \xrightarrow[k\to\infty]{} 0 \quad \forall j \geq 1$, i.e. $x_j^{(k)} \to y_j \quad \forall j \geq 1$. $\square$

Claim 2. ω is complete.

Proof. Assume $(x^{(k)})_{k=1}^{\infty}$ is Cauchy in ω, $x^{(k)} = (x_j^{(k)})_{j=1}^{\infty}$

That is, $d(x^{(k)}, x^{(\ell)}) \to 0$ as $k, \ell \to \infty$.

As above, this implies $|x_j^{(k)} - x_j^{(\ell)}| \to 0$ as $k, \ell \to \infty$

for each $j \geq 1$, i.e. $(x_j^{(k)})_{j=1}^{\infty}$ is Cauchy in $\mathbb{K}$.

Hence $\exists\, y_j = \lim_{k\to\infty} x_j^{(k)} \in \mathbb{K} \quad \forall j \geq 1$.

Claim 1 $\Rightarrow$ $x^{(k)} \to y = (y_j)$ in (ω, d). $\square$

Claim 3. ω is separable.

Proof. Write $Q = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$, $Q = \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$.

Let $D = \{ (q_1, q_2, \dots, q_n, 0, 0, \dots) \mid q_j \in Q \text{ for } j = 1, \dots, n \}$

"finite rational sequences"

$\Rightarrow$ $D \subset \omega$ is countable. We show: $\overline{D} = \omega$.

Let $x = (x_j) \in \omega$ be arbitrary.

For each j choose $q_j^{(1)}, q_j^{(2)}, \dots \in Q$

s.t. $q_j^{(k)} \longrightarrow x_j$ as $k \to \infty$.

Define

$$x^{(1)} = (q_1^{(1)}, 0, 0, \dots) \in D$$

$$x^{(2)} = (q_1^{(2)}, q_2^{(2)}, 0, 0, \dots) \in D$$

$$x^{(3)} = (q_1^{(3)}, q_2^{(3)}, q_3^{(3)}, 0, 0, \dots) \in D$$

etc ...

Then, by construction, $x_j^{(k)} = q_j^{(k)}$ for $k \geq j$

$\Rightarrow$ $x_j^{(k)} \xrightarrow[k \to \infty]{} x_j \quad \forall j \geq 1$

$\Rightarrow$ $x^{(k)} \longrightarrow x$ in (ω, d).

Hence $x \in \overline{D}$. $\square$