DYNAMICS OF LINEAR OPERATORS, AUTUMN 2010

Solutions to Exercises 1

(1.) (i) $\Leftrightarrow$ (ii) since $f^n(U) \cap V \neq \emptyset \Leftrightarrow U \cap f^{-n}(V) \neq \emptyset$

(i) $\Leftrightarrow$ (iv): f transitive $\Leftrightarrow$ for all nonempty open $U, V \subset X$

$\exists n \geq 0$ s.t. $f^n(U) \cap V \neq \emptyset$

$\Leftrightarrow$ for all nonempty open $U, V \subset X$: $\left(\bigcup_{n=0}^{\infty} f^n(U)\right) \cap V \neq \emptyset$

$\Leftrightarrow$ for all nonempty open $U \subset X$: $\bigcup_{n=0}^{\infty} f^n(U)$ is dense

(ii) $\Leftrightarrow$ (v): similar to (i) $\Leftrightarrow$ (iv)

(iii) $\Rightarrow$ (iv): Assume $U \subset X$ nonempty open

Let $A = \bigcup_{n=0}^{\infty} f^n(U)$

Then $f(A) \subset A$ and $\operatorname{int}(A) \supset U \neq \emptyset$

Hence (iii) $\Rightarrow$ must have $\bar{A} = X$, i.e. A is dense.

(i) $\Rightarrow$ (iii): Assume $A \subset X$ s.t. $f(A) \subset A$. Obs. that $f^n(A) \subset A$ $\forall n$

Let $U = \operatorname{int}(A)$, $V = \operatorname{int}(X \setminus A) = X \setminus \bar{A}$

$\Rightarrow$ open sets in X.

We have for all $n \geq 0$

$$f^n(U) \cap V \subset f^n(A) \cap (X \setminus A) \subset A \cap (X \setminus A) = \emptyset$$

Hence (i) $\Rightarrow$ $U = \emptyset$ or $V \neq \emptyset$. $\square$

(2.) a) Let $\alpha = m/n$ be rational. Then

$$(e^{2\pi i \alpha})^n = e^{2\pi i \alpha n} = e^{2\pi i m} = (e^{2\pi i})^m = 1^m = 1$$

Note that $f^k(e^{i\theta}) = (e^{2\pi i \alpha})^k e^{i\theta} = e^{2\pi i \alpha k} e^{i\theta}$ $\forall k \geq 0$

Hence if k is of the form $k = pn + j$

(for integers $p \geq 0$, $j = 0, 1, \ldots, n-1$), we get

$$f^k(e^{i\theta}) = e^{2\pi i\alpha(pn+j)}e^{i\theta} = (e^{2\pi i\alpha})^{np}e^{2\pi i\alpha j}e^{i\theta}$$
$$= 1\cdot e^{2\pi i\alpha j}e^{i\theta} = f^j(e^{i\theta})$$

Thus $\operatorname{orb}(f, e^{i\theta}) = \{f^j(e^{i\theta}) \mid j=0,1,\dots,n-1\}$

$$= \{e^{2\pi i\alpha j}e^{i\theta} \mid j=0,1,\dots,n-1\} \text{ finite.}$$

b) See Shapiro's lecture notes (link on the course www-page), Proposition 1.2.

3. First observe: If $f: X \to X$ transitive, then $\overline{f(X)} = X$.

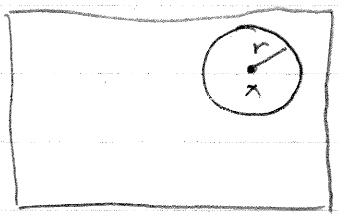
Reason: Obviously true if X consists of a single point.

Assume $\# X \geq 2$. If $\overline{f(X)} \neq X$, $\exists$ open ball $B(x,r) \subset X$ s.t. $f(X) \cap B(x,r) = \emptyset$.

Making $r>0$ smaller (if needed), we may assume that $\overline{B(x,r)} \neq X$, i.e. $\operatorname{int}(X \setminus B(x,r)) \neq \emptyset$.

Note that

$$f(X \setminus B(x,r)) \subset f(X) \subset X \setminus B(x,r)$$



X

$\Rightarrow$ condition (iii) of Prop. 2 fails for $A = X \setminus B(x,r)$

$\Rightarrow$ f not transitive, a contradiction! $\boxed{\text{obs.}}$

Actual proof: Assume to the contrary that for some U, V

$$\exists\, N = \max\{n \geq 0 \mid f^{-n}(U) \cap V \neq \emptyset\} = \max n(U,V)$$

By continuity $f^{-N}(U)$ is open (and $\neq \emptyset$ since $f^{-N}(U) \cap V \neq \emptyset$)

Also $f^{-(N+1)}(U) = f^{-1}(f^{-N}(U))$ is open, and observation above $\Rightarrow$ it is nonempty as well. Hence by transitivity

$$\exists\, n \geq 0 \text{ s.t. } f^{-(N+1+n)}(U) \cap V = f^{-n}(f^{-(N+1)}(U)) \cap V \neq \emptyset$$

$\Rightarrow$ $N+1+n \in n(U,V)$; a contradiction since $N+1+n > N$. $\square$

④ See e.g. [J. Väisälä, Topologia II, lause 10.8.]

⑤ a) <u>Observe</u> : $f \circ \varphi = \varphi \circ g \implies f^n \circ \varphi = \varphi \circ g^n \quad \forall n \geq 1$

(proof by induction!)

$$\begin{tikzcd}
U \arrow[d, phantom, "\cap"] & & V \arrow[d, phantom, "\cap"] \\
X \arrow[rr, "f^n"] & & X \\
Y \arrow[u, "\varphi"] \arrow[rr, "g^n"'] \arrow[d, phantom, "\cup"] & & Y \arrow[u, "\varphi"'] \arrow[d, phantom, "\cup"] \\
\varphi^{-1}(U) & & \varphi^{-1}(V)
\end{tikzcd}$$

Let $U, V \subset X$ be nonempty open sets. Since φ is continuous and has dense range, pre-images $\varphi^{-1}(U)$, $\varphi^{-1}(V)$ are nonempty open sets in Y.

g transitive $\implies \exists\, y \in \varphi^{-1}(U)$, $n \geq 0$ s.t. $g^n(y) \in \varphi^{-1}(V)$. Then $\varphi(y) \in U$ and

$f^n(\varphi(y)) = (f^n \circ \varphi)(y) = (\varphi \circ g^n)(y) = \varphi(g^n(y)) \in V$.

Thus $f^n(U) \cap V \neq \emptyset$. $\therefore$ f is transitive. $\square$

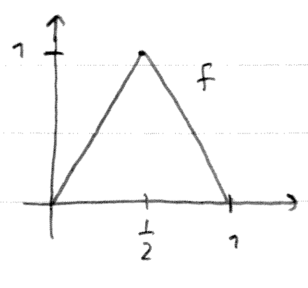
b) Let $\varphi(x) = \sin^2(\frac{\pi}{2}x)$, and let $f(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2-2x, & x \in [\frac{1}{2}, 1] \end{cases}$ <u>"tent map"</u>

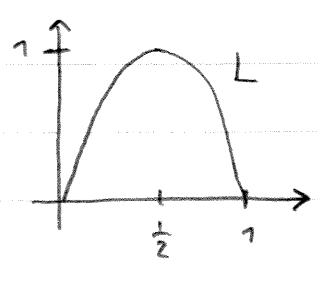
<u>Easy</u> : $\varphi : [0,1] \to [0,1]$ increasing homeomorphism

Compute for $x \in [0,1]$:

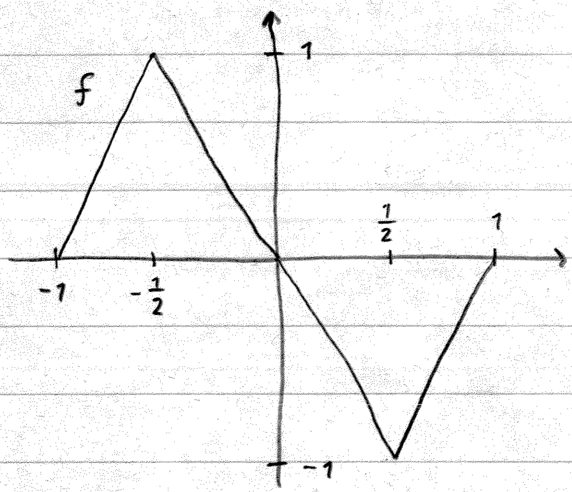
$$L(\varphi(x)) = 4 \sin^2(\tfrac{\pi}{2}x)\left(1 - \sin^2(\tfrac{\pi}{2}x)\right)$$
$$= 4 \sin^2(\tfrac{\pi}{2}x) \cos^2(\tfrac{\pi}{2}x) = \sin^2(2 \cdot \tfrac{\pi}{2}x)$$
$$= \sin^2(\pi x)$$

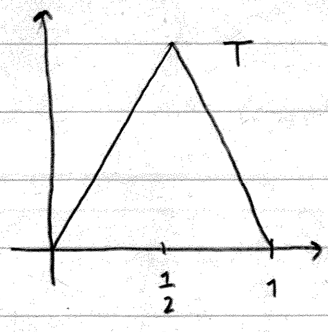
$$\varphi(f(x)) = \begin{cases} \sin^2(\frac{\pi}{2} \cdot 2x) = \sin^2(\pi x), & x \in [0, \frac{1}{2}] \\ \sin^2(\frac{\pi}{2}(2-2x)) = \sin^2(\pi - \pi x), & x \in [\frac{1}{2}, 1] \end{cases}$$
$$= \sin^2(\pi x) \qquad \therefore \quad L \circ \varphi = \varphi \circ f$$





6.





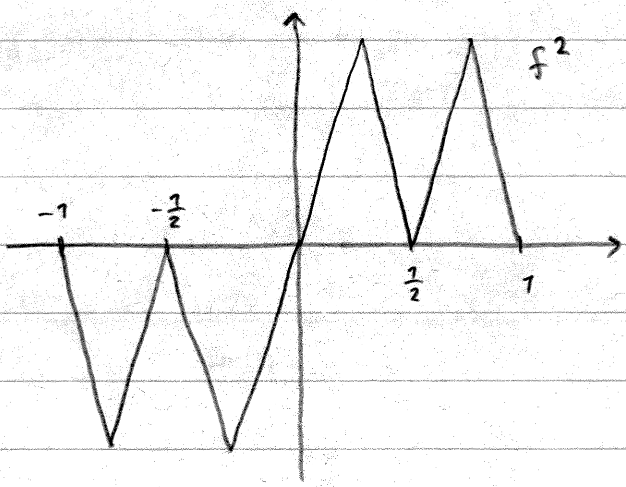
tent map

$$T(x) = \begin{cases} 2x , & x \in [0, \frac{1}{2}] \\ 2-2x , & x \in [\frac{1}{2}, 1] \end{cases}$$

Observe:
$$f(x) = \begin{cases} T(-x) , & x \in [-1, 0] \\ -T(x) , & x \in [0, 1] \end{cases}$$

$\Rightarrow$

$$f^2(x) = \begin{cases} -T(T(-x)) = -T^2(-x) , & x \in [-1, 0] \\ T(-(-T(x))) = T^2(x) , & x \in [0, 1] \end{cases}$$



(see page 5 in lecture notes for T^2)

Note that $f^2([0,1]) = [0,1]$, where $\mathrm{int}([0,1]) \neq \emptyset$ and $\mathrm{int}([-1,1] \setminus [0,1]) \neq \emptyset$

$\Rightarrow$ (iii) of Prop. 2 fails for f^2 $\Rightarrow$ f^2 non-transitive !

Claim: f is transitive

By induction (check it!) the above formulas for f, f^2 generalize to the following:

$$(*) \quad f^n(x) = \begin{cases} T^n(-x), & x \in [-1,0] \\ -T^n(x), & x \in [0,1] \end{cases} \quad \text{when } n \text{ is } \underline{\text{odd}}$$

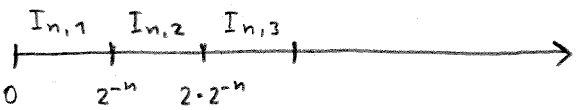
$$(**) \quad f^n(x) = \begin{cases} -T^n(-x), & x \in [-1,0] \\ T^n(x), & x \in [0,1] \end{cases} \quad \text{when } n \text{ is } \underline{\text{even}}$$

<u>Fact:</u> T satisfies $T^m(I_{n,k}) = [0,1]$ $\forall\, m \geq n$, $k = 1, 2, \dots, 2^n$

where

$$I_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right] \quad \text{(dyadic interval at level } n\text{)}$$

(see pages 5-6 in lecture notes).



Assume $U \subset [-1,1]$ nonempty open set.

Then U contains

a) an interval $I_{n,k}$ (for large enough n and suitable k)

<u>or</u> b) an interval $-I_{n,k} = \left[-\frac{k}{2^n}, -\frac{k-1}{2^n}\right]$

We may assume n is <u>even</u>. Then:

<u>case a)</u>: $(**) \Rightarrow f^n(I_{n,k}) = T^n(I_{n,k}) \overset{\text{fact}}{=} [0,1]$

$(*) \Rightarrow f^{n+1}(I_{n,k}) = -T^{n+1}(I_{n,k}) = -[0,1] = [-1,0]$

Hence $f^n(U) \cup f^{n+1}(U) = [0,1] \cup [-1,0] = [-1,1]$

Prop. 2 (iv) $\Rightarrow$ f <u>transitive</u>.

<u>case b)</u>: $(**) \Rightarrow f^n(-I_{n,k}) = -T^n(I_{n,k}) \overset{\text{fact}}{=} -[0,1] = [-1,0]$

$(*) \Rightarrow f^{n+1}(-I_{n,k}) = T^{n+1}(I_{n,k}) = [0,1]$

Hence $f^n(U) \cup f^{n+1}(U) = [-1,0] \cup [0,1] = [-1,1]$

Prop. 2 (iv) $\Rightarrow$ f <u>transitive</u>. $\square$