Dynamics of linear operators – Exercise set 6 (15. 12. 2010)

1. Recall the notion of quasi-conjugacy from Exercise 1.5. Show that chaoticity is preserved under quasi-conjugacy: if $g\colon Y \to Y$ is chaotic and $f\colon X \to X$ is quasi-conjugate to g, then f is chaotic. Here X and Y are assumed to be metric spaces without isolated points, and f and g are continuous.

2. Consider Birkhoff's operator $T\colon H(\mathbb{C}) \to H(\mathbb{C})$, $Tf(z) = f(z+1)$ (see Example 2.13 and Exercise 3.2). Prove that T is chaotic.

   [*Hint.* To verify the density of periodic points note that for $f(z) = e^{\lambda z}$ one has $Tf(z) = e^{\lambda} f(z)$ and follow the idea used for MacLane's operator. When is e^{λ} a root of unity?]

3. Consider the weighted backward shift $T\colon \ell^1 \to \ell^1$,

   $$T(x_1, x_2, x_3, \ldots) = (2x_2, \tfrac{3}{2}x_3, \tfrac{4}{3}x_4, \tfrac{5}{4}x_5, \ldots).$$

   (see Exercises 3.3 and 3.4). Prove that T is hypercyclic but its only periodic point is the zero sequence $\bar{0}$. Therefore T is not chaotic.

   [*Hint.* Assume to the contrary that there is a nonzero $x \in \ell^1$ such that $T^n x = x$. Try to get a contradiction with the fact that $\|x\|_1 = \sum_k |x_k| < \infty$.]

4. Let X be a separable Banach space and $T \in L(X)$. Prove that the following conditions are equivalent:

   i) T has sensitive dependence on initial conditions.
   ii) $\sup_n \|T^n\| = \infty$.
   iii) T has an unbounded orbit, i.e. $\sup_n \|T^n x\| = \infty$ for some $x \in X$.

   [*Hints.* Apply the tricks used in the proof of Proposition 5.9. Also recall the Banach–Steinhaus theorem (a.k.a. the principle of uniform boundedness).]

5. Let X be a complex vector space and $T\colon X \to X$ linear. Show that the set of periodic points of T is given by

   $$\mathrm{per}(T) = \mathrm{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } \lambda^n = 1 \text{ for some } n \geq 1\}.$$

   [*Outline.* For the inclusion $\subset$ suppose that $T^n x = x$. Write $z^n - 1 = (z - \lambda_1) \cdots (z - \lambda_n)$ where $\lambda_k = e^{2\pi i k/n}$ are the n:th roots of unity. Argue that $1 = \sum_{k=1}^n a_k p_k(z)$ for some $a_k \in \mathbb{C}$ where $p_k(z) = \prod_{j \neq k} (z - \lambda_j)$. Writing $y_k = p_k(T)x$ we then have $x = \sum_{k=1}^n a_k y_k$. Note that $(T - \lambda_k I) y_k = 0$.]

Dictionary: root of unity = ykkösenjuuri, sensitive dependence on initial conditions = alkuarvoherkkä

Last lectures will be held on **Mon 13. 12**.

Last exercise session will be held on **Wed 15. 12. at 14.15**. After the exercise session (around 16.00), coffee/tea and cake will be served. Everyone is welcome!