Dynamics of linear operators – Exercise set 4 (1. 12. 2010)

Note: In the following exercises, X denotes a separable Banach space (or, more generally, a separable Fréchet space).

1. Recall that $T \in L(X)$ is of *finite rank* if its range $T(X)$ is finite-dimensional. Show that a finite rank operator is never hypercyclic.

   [*Recall.* We have shown that operators on finite-dimensional spaces are never hypercyclic.]

2. Assume that $\lambda \in \mathbb{K}$ satisfies $\lambda^p = 1$ for some integer $p \geq 1$. Show that if T is hypercyclic, then λT is hypercyclic. In fact, $HC(\lambda T) = HC(T)$. [*Hint.* Ansari's theorem.]

   [It can be shown that the result is true for every λ with $|\lambda| = 1$.]

3. Assume that $T \in L(X)$ is hypercyclic. Show that $HC(T)$ is a connected set.

   [*Hints.* Exercise 3.5 and Lemma 3.2.]

4. Let $T \in L(X)$ and suppose that there are vectors $x_1, x_2, \ldots \in X$ such that

   $$\bigcup_{j=1}^{\infty} \overline{\mathrm{orb}(T, x_j)} = X.$$

   Prove that $x_j \in HC(T)$ for some j. In particular, T is hypercyclic. [*Magic.* Baire.]

5. An operator $T \in L(X)$ is *supercyclic* provided there exists a vector $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{K},\ n \geq 0\}$ is dense in X.

   Prove that for any $T \in L(X)$ the following conditions are equivalent:

   (i) T is supercyclic.
   (ii) For every pair $U, V \subset X$ of nonempty open sets there exist $\lambda \in \mathbb{K}$ and $n \geq 0$ such that $\lambda T^n(U) \cap V \neq \emptyset$.

   (This is a "supercyclic" analogue of Birkhoff's transitivity theorem.)

6. Give examples of supercyclic linear operators on $\mathbb{R}$ and $\mathbb{R}^2$. (Here $\mathbb{K} = \mathbb{R}$.)

Dictionary: (of) finite rank = äärellisasteinen