Dynamics of linear operators – Exercise set 3 (24. 11. 2010)

1. Prove the following variant of Kitai's criterion:

   Let X be a separable Banach (or Fréchet) space and $T \in L(X)$. Suppose that there are dense sets $X_0, Y_0 \subset X$ with the following properties:

   i) $T^n x \to 0$ for each $x \in X_0$,
   ii) for each $y \in Y_0$ there is a sequence (u_n) in X such that $u_n \to 0$ and $T^n u_n \to y$.

   Then T is hypercyclic.

2. Use Kitai's criterion to show that Birkhoff's operator $T\colon H(\mathbb{C}) \to H(\mathbb{C})$, $Tf(z) = f(z+1)$, is hypercyclic.

   [*Hints.* Let $X_0 = Y_0$ be the set of functions of the form $f_{p,\alpha,k}(z) = p(z)\exp\{-\alpha(z-k)^2\}$, where p is a polynomial, $\alpha > 0$ and $k \geq 0$ is an integer. To verify density, note that every polynomial p can be approximated by $f_{p,\alpha,k}$ uniformly on compacts as $\alpha \to 0$.]

3. The *weighted (backward) shift* operator B_a acting on scalar sequences is defined as

   $$B_a(x_1, x_2, x_3, \ldots) = (a_2 x_2, a_3 x_3, a_4 x_4, \ldots),$$

   where the *weight sequence* $a = (a_k)$ consists of *non-zero* scalars. (Note that the value of a_1 is irrelevant.)

   Consider B_a as an operator on $\omega = \mathbb{K}^{\mathbb{N}}$, the Fréchet space of all scalar sequences. Show that B_a is a) always continuous, and b) always hypercyclic (assuming only that $a_k \neq 0$ for each k)!

   [*Hint.* For part (b) recall that the space c_{00} of "finite sequences" is dense in ω.]

4. Let $a = (a_k)$ be a bounded weight sequence. Then $B_a \in L(\ell^p)$ for $1 \leq p < \infty$. Prove that B_a is hypercyclic on ℓ^p if $\sup_{n\geq 1} |a_1 a_2 \cdots a_n| = \infty$.

   [*Hint.* Apply the Hypercyclicity criterion (Theorem 2.7) with $S_{n_k} = S^{n_k}$ where S is a weighted forward shift.]

5. Let $T \in L(X)$ be hypercyclic on a Banach space X. Given a hypercyclic vector $x \in HC(T)$, define

   $$M = \{p(T)x : p \text{ is a polynomial}\} = \operatorname{span}(\operatorname{orb}(T, x)).$$

   Show that every vector $y \in M \setminus \{0\}$ is hypercyclic for T. Thus, every hypercyclic operator admits a (dense) linear subspace consisting, except for zero, of hypercyclic vectors!

   [*Hint.* Lemma 3.1 shows that $p(T)$ has a dense range in X for every non-zero p.]

Dictionary: dense range = tiheä kuva(joukko)