Dynamics of linear operators – Exercise set 2 (17. 11. 2010)

1. Let X be a Banach space and $T \in L(X)$ a contraction, i.e. $\|T\| \leq 1$. Show that T is not hypercyclic.

2. Recall that ℓ^∞ is the Banach space consisting of all *bounded* scalar sequences $x = (x_k)$ under the supremum norm $\|x\|_\infty = \sup_k |x_k|$. Show that the Rolewicz operator $T = 2B : (x_1, x_2, \ldots) \mapsto (2x_2, 2x_3, \ldots)$ is non-transitive on ℓ^∞.

   [*Note.* ℓ^∞ is non-separable. Therefore it supports no hypercyclic operators and Birkhoff's transitivity theorem is not available.]

3. Let X be Banach space and $T \in L(X)$. Assume that the adjoint T^* has an eigenvalue, i.e. there exist a scalar $\alpha \in \mathbb{K}$ and a non-zero functional $\varphi \in X^*$ such that $T^*\varphi = \alpha\varphi$. Show that T is not hypercyclic.

   [*Hints.* Recall that the adjoint (or transpose) $T^* \in L(X^*)$ is defined by $(T^*\varphi)(x) = \varphi(Tx)$, i.e. $\langle T^*\varphi, x\rangle = \langle \varphi, Tx\rangle$ for all $x \in X$ and $\varphi \in X^*$, where X^* is the dual of X.

   Assume to the contrary that T has a hypercyclic vector x, and iterate the eigenvalue equation $T^*\varphi = \alpha\varphi$ at x to reach a contradiction.]

4. Suppose that $T \in L(X)$ is hypercyclic on a Banach (or Fréchet) space X. Show that every $z \in X$ is the sum $z = x + y$ of two hypercyclic vectors x and y; that is, we may write $X = HC(T) + HC(T)$.

   [*Hint.* It is enough to show that $HC(T)$ and $z - HC(T)$ have a non-empty intersection. Recall Baire's theorem.]

5. Let X be a vector space and $(p_n)_{n=1}^\infty$ an increasing and separating sequence of seminorms on X. Define (as in the lecture notes)

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1,\ p_n(x-y)\}, \qquad x, y \in X.$$

   Prove:

   a) d is a metric on X which is translation-invariant (i.e. $d(x,y) = d(x+z, y+z)$ for all $x, y, z \in X$).
   b) A sequence (x_k) of vectors in X converges to a vector $y \in X$ with respect to d if and only if $\lim_{k\to\infty} p_n(x_k - y) = 0$ for each n.

6. Let $\omega = \mathbb{K}^{\mathbb{N}}$ be the vector space of *all* scalar sequences:

$$\omega = \{(x_k)_{k=1}^\infty : x_k \in \mathbb{K} \text{ for all } k\}.$$

   For $n \geq 1$ define

$$p_n(x) = \sup\{|x_k| : 1 \leq k \leq n\}, \qquad x = (x_k) \in \omega.$$

   Observe that (p_n) is an increasing and separating family of seminorms on ω. Show that if d is the metric defined in the previous exercise, then the convergence with respect to d just means pointwise convergence. Is ω complete? Is it separable?

Dictionary: adjoint = adjungaatti, eigenvalue = ominaisarvo