Dynamics of linear operators – Exercise set 1 (10. 11. 2010)

1. Prove Proposition 2 (page 4 of lecture notes): If X is a metric space and $f\colon X \to X$, then the following conditions are equivalent:

   i) f is transitive.
   ii) For any nonempty open sets $U, V \subset X$ there exists $n \geq 0$ such that $f^{-n}(U) \cap V \neq \emptyset$.
   iii) If $A \subset X$ satisfies $f(A) \subset A$, then $\mathrm{int}(A) = \emptyset$ or $\overline{A} = X$.
   iv) For any nonempty open set $U \subset X$ the union $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in X.
   v) For any nonempty open set $U \subset X$ the union $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is dense in X.

2. Let $S = \{e^{i\theta} : \theta \in \mathbb{R}\}$ be the unit circle in the complex plane. Let $\alpha > 0$ and define $f\colon S \to S$ by $f(e^{i\theta}) = e^{2\pi i\alpha}e^{i\theta} = e^{i(2\pi\alpha+\theta)}$ (rotation through the angle $2\pi\alpha$). Prove:

   a) If α is rational, then the orbit of every point of S under f is finite.
   b) If α is irrational, then the orbit of every point of S under f is dense.

   [*Recall.* $e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$ for all $\theta \in \mathbb{R}$.]

3. Suppose that X is a metric space and $f\colon X \to X$ is continuous. Given two nonempty open sets $U, V \subset X$, define the *hitting time set*

   $$n(U, V) = \{n \geq 0 : f^{-n}(U) \cap V \neq \emptyset\} = \{n \geq 0 : U \cap f^n(V) \neq \emptyset\}.$$

   Show that if f is transitive, then $n(U, V)$ is infinite.

4. Prove Baire's theorem: If X is a complete metric space and $U_1, U_2, \ldots$ are open and dense subsets of X, then the intersection $A = \bigcap_{k=1}^{\infty} U_k$ is dense.

   [Consult the material of Topology II or Functional Analysis (or other literature) if needed.]

5. Let X and Y be metric spaces, and consider two maps $f\colon X \to X$ and $g\colon Y \to Y$. We say that f is *quasi-conjugate* to g if there exists a continuous map $\varphi\colon Y \to X$ such that $\varphi(Y)$ is dense in X and $f \circ \varphi = \varphi \circ g$. If φ can be chosen to be a homeomorphism, then f and g are *conjugate*.

   a) Show that topological transitivity is preserved under quasi-conjugacy: if g is transitive and f is quasi-conjugate to g, then f is transitive. [*Hint.* $f^n \circ \varphi = \varphi \circ g^n$.]
   b) Show that the *logistic map* $L\colon [0, 1] \to [0, 1]$, $L(x) = 4x(1 - x)$, is conjugate to the tent map (page 5 of lecture notes) and hence transitive.

6. Define the *double tent map* $f\colon [-1, 1] \to [-1, 1]$ by

   $$f(x) = \begin{cases} 2 + 2x, & \text{if } -1 \leq x \leq -\frac{1}{2}, \\ -2x, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ -2 + 2x, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

   Draw the graphs of f and $f^2 = f \circ f$. Show that f is transitive but f^2 is not.

   [*Hints.* It is easy to see that f^2 is nontransitive. To show that f is transitive, try to interpret it as consisting of two copies of the ordinary tent map dynamics such that every iteration step alternates from one copy into the other.]

Dictionary: orbit = rata, isolated point = erakkopiste, uncountable = ylinumeroituva, hitting time set = osumahetkien joukko