The logic of uninformative improper prior for σ^2

First, consider the location parameter: if the density $\pi(x - \mu \mid \mu)$ is a function, $g(u)$ where $u = x - \mu$, that is free of μ and x, then $x - \mu$ is a pivotal quantity and μ is called a pure location parameter. (For example, $x \sim \mathrm{N}(\mu, \sigma^2)$, then $u = x - \mu \sim \mathrm{N}(0, \sigma^2) = g(u)$). Then, 'a reasonable' uninformative prior for μ would be such that the posterior $\pi(x - \mu \mid x)$ would be $g(x - \mu)$ which is free of both x and μ. From Bayes formula we get: $\pi(x - \mu \mid x) \propto \pi(\mu)\pi(x - \mu \mid \mu)$, which means that $\pi(\mu)$ has to be constant over the range $-\infty, \infty$.

Second, consider similarly the scale parameter: if the density $\pi(x/\theta \mid \theta)$ is a function, $g(u)$ where $u = x/\theta$, that is free of θ and x, then x/θ is a pivotal quantity and θ is called a pure scale parameter. (For example, $x \sim \mathrm{N}(0, \sigma^2)$, then $u = x/\sigma \sim \mathrm{N}(0, 1) = g(u)$). Then, 'a reasonable' uninformative prior for θ would be such that the posterior $\pi(x/\theta \mid x)$ would be $g(x/\theta)$ which is free of both x and θ. By transformation of variables: $\pi_x(x \mid \theta) = \frac{1}{\theta}\pi_u(u \mid \theta)$. Likewise: $\pi_\theta(\theta \mid x) = \frac{x}{\theta^2}\pi_u(u \mid x)$. Now, recall that $\pi(u \mid \theta)$ and $\pi(u \mid x)$ were both the same as $g(u)$. Hence, we get $\pi(\theta \mid x) = \frac{x}{\theta}\pi(x \mid \theta)$. Therefore, the prior is $\pi(\theta) \propto 1/\theta$, or equivalently $\pi(\theta^2) \propto 1/\theta^2$, or $\pi(\log(\theta)) \propto 1$.

Ref: Gelman et al: Bayesian Data Analysis.