6. Conformal invariants

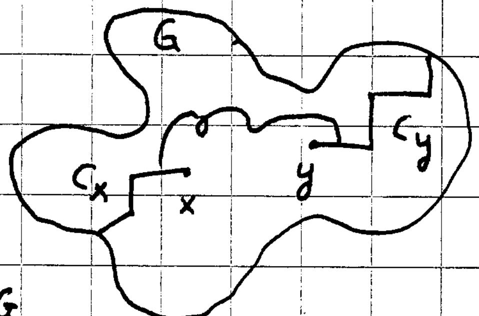
6.1. <u>What comes next?</u> One of the main goals will be to explore connections between conformal invariants and metrics applying the results from Section 5. For a domain $G \subsetneq \overline{\mathbb{R}}^n$ and $x, y \in G$ we will define $\lambda_G(x,y)$ and its "dual" counterpart $\mu_G(x,y)$ which both measure "the relative distance from x to y w.r.t. G". Both λ_G and μ_G will be defined in terms of moduli of curve families implying that they will be Möbius invariant. If $G = B^n$ both can be expressed in terms of τ_n and ϱ_{B^n} which is the ideal case (because ϱ_{B^n} is conf. invariant). In a general domain G let

$$r_G(x,y) = |x-y| / \min\{d(x,\partial G), d(y,\partial G)\}.$$

Clearly r_G is invar. under similarity maps (it can be shown that it is not Möbius invar.). For a large class of domains we may estimate λ_G in terms of r_G.

6.2. <u>Invariants λ_G and μ_G.</u> Let $G \subsetneq \overline{\mathbb{R}}^n$ be a domain, $x, y \in G$, $x \neq y$,

(6.3) $\quad \lambda_G(x,y) = \inf M(\Delta(C_x, C_y; G))$

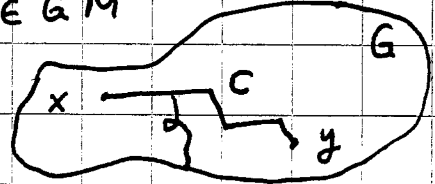


where the inf is taken over all pairs of sets C_x, C_y such that $C_z = \gamma_z[0,1)$ and $\gamma_z : [0,1) \to G$ is a curve such that $\gamma_z(0) = z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$. Obviously λ_G is a conf. invar. i.e.

$$\lambda_G(x,y) = \lambda_{fG}(fx, fy) \quad \forall\, f \in GM$$

For $x, y \in G$ set

(6.4) $\quad \mu_G(x,y) = \inf_C M(\Delta(C, \partial G; G))$



where the inf is taken over all connected sets C, $x, y \in C$

6.5. <u>Rmk.</u> Below we will assume $\mathrm{card}(\overline{\mathbb{R}}^n \setminus G) \geq 2$ to avoid trivial cases. Clearly also μ_G is a conf. invar. It is not difficult to show that μ_G is a metric if $\mathrm{cap}(\partial G) > 0$.

6.6. Rmk. Let $D \subset G$ be a domain. Then for $a, b \in D$, $a \neq b$, we have $\mu_D(a,b) \geq \mu_G(a,b)$ and $\lambda_G(a,b) \geq \lambda_D(a,b)$. J. Ferrand proved ca 1996 that $\lambda_D(a,b)^{1/(1-n)}$ is a metric, thus answering a question posed in CGQM,

6.7. Lemma. For $x, y \in B^n$, $x \neq y$

(1) $\mu_{B^n}(x,y) = \gamma(1/\mathrm{th}(\varrho(x,y)/2)) = 2^{n-1}\tau(1/\mathrm{sh}^2(\varrho(x,y)/2))$,

(2) $\lambda_{B^n}(x,y) = \frac{1}{2}\tau(\mathrm{sh}^2(\varrho(x,y)/2)) = 2^{-n}\gamma(\mathrm{ch}(\varrho(x,y)/2))$.

Pf. (1) follows from 5.33 and 5.20

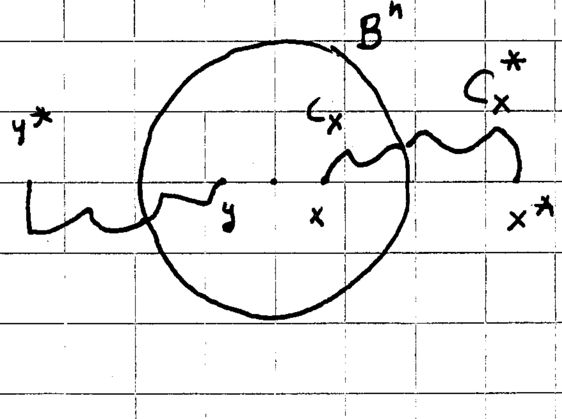
(2) The assertion is $GM(B^n)$-invar., hence we may assume that $x = re_1 = -y$ and $r = \mathrm{th}(\varrho(x,y)/4)$ (cf.). By

$$\lambda_{B^n}(x,y) \leq M(\Delta(E, -E; B^n)) \qquad (E = [re_1, e_1))$$
$$\leq \frac{1}{2} M(\Delta(E_2, -E_2; \mathbb{R}^n)) \qquad (E_2 = [re_1, \tfrac{1}{r}e_1])$$
$$= \frac{1}{2}\tau\left(\frac{4r^2}{(1-r^2)^2}\right) = \frac{1}{2}\tau\left(\mathrm{sh}^2\frac{\varrho(x,y)}{2}\right).$$

N.B. $\frac{4r^2}{(1-r^2)^2} = \left(\frac{2r}{1-r^2}\right)^2 = \left(\frac{2\,\mathrm{th}\frac{\varrho}{4}}{1-\mathrm{th}^2\frac{\varrho}{4}}\right)^2 = \left(2\,\mathrm{sh}\frac{\varrho}{4}\,\mathrm{ch}\frac{\varrho}{4}\right)^2 = \mathrm{sh}^2\frac{\varrho(x,y)}{2}$

Therefore it is enough to prove $\lambda_{B^n} \geq \frac{1}{2}\tau(\mathrm{sh}^2\frac{\varrho}{2})$. Let C_x, C_y be as in (6.3) and C_x^*, C_y^* their images under the inversion $x \mapsto x/|x|^2$. $C_x^s = C_x \cup C_x^*$, $C_y^s = C_y \cup C_y^*$. May ass. $C_y \not\ni 0$.

Choose compact connected sets $E \subset C_x$, $x \in E$, $F \subset C_y$, $y \in F$ and let $E^s = E \cup E^*$, $F^s = F \cup F^*$. Let $\mathrm{Sym}(E^s)$ E^s: symmetrized in the posit. x_1-axis and $\mathrm{Sym}(F^s)$ F^s symm. in the negat x_1-axis



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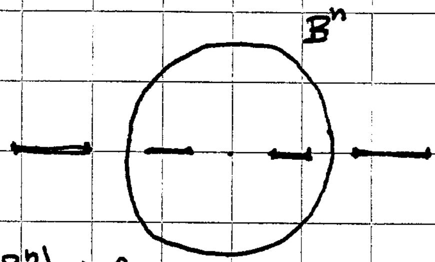
5.13 $\Rightarrow$

$$(*)\begin{cases} \operatorname{cap}(\mathbb{R}^n \setminus E^s, F^s) \geq \operatorname{cap}(\mathbb{R}^n \setminus \operatorname{Sym}(E^s), \operatorname{Sym}(F^s)) \\ \longrightarrow M(\Delta([-\tfrac{1}{r}e_1, -re_1], [re_1, \tfrac{1}{r}e_1])) \\ = \tau\left(\frac{4r^2}{(1-r^2)^2}\right) = \tau\left(\operatorname{sh}^2 \frac{\varrho(x,y)}{2}\right) \end{cases}$$



The above convergence means: When $d(E, \partial B^n) \to 0$ and $d(F, \partial B^n) \to 0$, then $\operatorname{Sym}(E^s) \to [re_1, \frac{1}{r}e_1]$, $\operatorname{Sym}(F^s) \to [-\frac{1}{r}e_1, -re_1]$ and $(*)$ holds. On the other hand by

$$M(\Delta(C_x, C_y; B^n)) \geq M(\Delta(E, F; B^n))$$
$$\geq \frac{1}{2}\operatorname{cap}(\mathbb{R}^n \setminus E^s, F^s) \to \frac{1}{2}\tau\left(\operatorname{sh}^2 \frac{\varrho(x,y)}{2}\right)$$

when $d(E, \partial B^n) \to 0$, $d(F, \partial B^n) \to 0$ and $x \in E$, E continuum $y \in F$, F contin.

Because C_x, C_y were arbitrary sets as in (6.3), the assertion $\lambda_{B^n} \geq \frac{1}{2}\tau(\operatorname{sh}^2 \frac{\varrho}{2})$ follows.

6.8. <u>Exer.</u> Apply 5.25(3) to prove that

$$\frac{1}{2}\tau\left(\operatorname{sh}^2(\varrho(x,y)/2)\right) \geq -c_n \log \operatorname{th}(\varrho(x,y)/4)$$
$$= 2c_n \operatorname{arth}(e^{-\varrho(x,y)/2}) \geq 2c_n e^{-\varrho(x,y)/2}$$

(Hint: Show first that $2\operatorname{ch}^2 A = 1 + \operatorname{ch} 2A$ and $\operatorname{sh} 2A = 2 \operatorname{sh} A \operatorname{ch} A$)

Apply 5.25(3) also to prove that

$$\frac{1}{2}\tau\left(\operatorname{sh}^2 \frac{\varrho(x,y)}{2}\right) \leq \frac{c_n}{2}\,\mu\left(\operatorname{th}^2(\varrho(x,y)/4)\right)$$

$$< \frac{c_n}{2} \log \frac{4}{\operatorname{th}^2 \frac{\varrho(x,y)}{4}} = c_n \log\left(2/\operatorname{th}(\varrho(x,y)/4)\right).$$

6.9. Lemma Let $G \subsetneq \mathbb{R}^n$ be a domain, $x \in G$, $y \in B^n(x, d(x)) = B_x$, $y \neq x$. Then

(1) $$\lambda_G(x,y) \geq \lambda_{B_x}(x,y) \geq c_n \log \frac{d(x)}{|x-y|},$$

(2) $$\mu_G(x,y) \leq \mu_{B_x}(x,y) = \gamma(d(x)/|x-y|) \leq \omega_{n-1} \left(\log \frac{d(x)}{|x-y|}\right)^{1-n}.$$

Pf (1) The spherical cap-inequality gives

$$M(\Delta(C_x, C_y; G)) \geq M(\Delta(C_x, C_y; B_x)) \geq \lambda_{B_x}(x,y)$$
$$\geq c_n \log(d(x)/|x-y|)$$

whenever C_x, C_y are as in (6.3). [A 2nd pf: Apply 6.8].

(2) 5.33 & (5.31) $\Rightarrow$

$$\mu_G(x,y) \leq \mu_{B_x}(x,y) = M(\Delta([x,y], \partial B_x)) = \gamma(d(x)/|x-y|)$$
$$\leq \omega_{n-1} (\log(d(x)/|x-y|))^{1-n}.$$

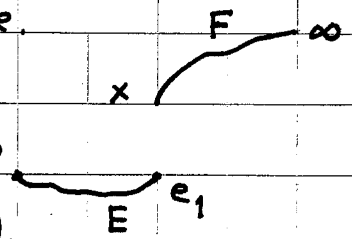
6.10. The function $p(x)$. Fix $x \in \mathbb{R}^n \setminus \{0, e_1\}$ and set

(6.11) $$p(x) = \inf_{E,F} M(\Delta(E,F))$$

where the inf is taken over all pairs of continua E, F such that $0, e_1 \in E$, $x, \infty \in F$.

Teichmüller's problem: Find $p(x)$ in terms of well-known functions.

This problem was presented in 1938 and solved by Schiffer in 1948 for $n=2$. We consider here the n-dim. case.



6.12. Lemma. For $x \in \mathbb{R}^n \setminus \{0, e_1\}$

$$p(x) \geq \max\{\tau(|x|), \tau(|x-e_1|)\}$$

with equality if $x = se_1$ and $s \in (-\infty, 0) \cup (1, \infty)$.

Pf. Spher. symm. with center at $0 \Rightarrow p(x) \geq \tau(|x|)$

$e_1 \Rightarrow p(x) \geq \tau(|x-e_1|)$

If $x = te_1$, $t > 1$ and $E_0 = [0, e_1]$, $F_0 = [te_1, \infty)$ $p(x) = \tau(t-1)$

For the case $x = -te_1$, $t>0$ the choice $E_1 = [0, e_1]$, $F_1 = [-te_1, \infty)$ yields $p(x) = M(\Delta(E_1, F_1)) = \tau(t)$.

We next map the quadruple $(0, e_1, x, \infty)$ to $(-e_1, y, -y, e_1)$.

6.13. Lemma. Let $f \in \mathcal{GM}$ with $(0, e_1, x, \infty) \xmapsto{f} (-e_1, y, -y, e_1)$, $|y| \le 1$. Then

$$|y| = \frac{|x-e_1|}{1+|x|+t} \quad \text{and} \quad |y+e_1|^2 = \frac{|y-e_1|^2}{|x|} = \frac{4}{1+|x|+t}$$

where $t = ((1+|x|)^2 - |x-e_1|^2)^{1/2}$.

Pf. The Möbius invariance yields

$|0, e_1, x, \infty| = |-e_1, y, -y, e_1|$ and $|0, e_1, \infty, x| = |-e_1, y, e_1, -y|$,

and, equivalently,

$$|y-e_1|^2 = |x|\,|y+e_1|^2 \quad \text{and} \quad 4|y| = |x-e_1|\,|y+e_1|^2 .$$

The first equation implies

$$2\, y \cdot e_1 = \frac{1-|x|}{1+|x|}(1+|y|^2)$$

Substitution of this into the second equation yields

$$4|y| = |x-e_1|\left(|y|^2 + \frac{1-|x|}{1+|x|}(1+|y|^2) + 1\right) \Leftrightarrow$$

$$|y|^2 - 2|y|\frac{1+|x|}{|x-e_1|} + 1 = 0 \quad \Rightarrow \quad |y| = \frac{1+|x| \pm t}{|x-e_1|}$$

The $-$sign yields $|y| \le 1$ and the desired formula follows. This computation also yields the desired formula for $|y+e_1|$.

6.14. Cor. Let $f \in \mathcal{GM}$ with $(a,b,c,d) \mapsto (-e_1, y, -y, e_1)$, $|y| \le 1$. If $r = |b,a,c,d|$, $s = |a,b,c,d|$, $t = \sqrt{(1+s)^2 - r^2}$ then

$$|y| = \frac{r}{1+s+t} \quad \text{and} \quad |y+e_1|^2 = \frac{|y-e_1|^2}{s} = \frac{4}{1+s+t} .$$

6.15. Lemma. For $a \in (0,1)$ let $b = \frac{2a}{1+a^2}$. Then for $r > 0$

$$M(\Delta([-are_1, are_1], S^{n-1}(r))) = M(\Delta[0, bre_1], S^{n-1}(r))) = \gamma\left(\frac{1+a^2}{2a}\right)$$

Pf. Choose $h \in \mathcal{GM}(B^n(r))$ such that $(-re_1, -are_1, are_1, re_1)$ $\mapsto (-re_1, 0, bre_1, re_1)$. Then

$$|-re_1, -are_1, are_1, re_1| = |-re_1, 0, bre_1, re_1| .$$

This implies $b = 2a/(1+a^2)$. The equalities follow from the conf. inv. of modulus and the def. of γ.

6.16. Lemma. Let $y \in B^n \setminus \{0\}$, $E = [-y, y]$, $F = [-e_1, \infty) \cup [e_1, \infty)$ $E_1 = [-|y|e_1, |y|e_1]$. Then

$$M(\Delta(E,F)) \le M(\Delta(E_1,F)) = \tau\left(\frac{(1-|y|)^2}{4|y|}\right).$$

Pf. By the def. of τ and the conf. invariance of M we have

$$M(\Delta(E_1,F)) = \tau(||y|e_1, -|y|e_1, e_1, -e_1|) = \tau\left(\frac{(1-|y|)^2}{4|y|}\right).$$

To prove the ineq. write $y = t^2 e$, $|e| = 1$, $t \in (0,1)$, $S = S^{n-1}(t)$ $\Gamma_1 = \Delta(E,S)$, $\Gamma_2 = \Delta(F,S)$. Then by Lemma 6.15

$$M(\Gamma_1) = M(\Gamma_2) = \gamma\left(\frac{1+t^2}{2t}\right) = 2^{n-1}\tau\left(\left(\frac{1+|y|}{2\sqrt{|y|}}\right)^2 - 1\right) = 2^{n-1}\tau\left(\frac{(1-|y|)^2}{4|y|}\right).$$

By 3.27

$$M(\Delta(E,F))^{1/(1-n)} \ge M(\Gamma_1)^{1/(1-n)} + M(\Gamma_2)^{1/(1-n)} = 2\,\tfrac{1}{2}\,\tau\left(\frac{(1-|y|)^2}{4|y|}\right)^{1/(1-n)}$$

which yields the ineq.

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6.17. Theorem. For $x \in \mathbb{R}^n \setminus \{0, e_1\}$ there exists a circular arc E with $0, e_1 \in E$ and a ray F with $x \in F$ such that

$$p(x) \le M(\Delta(E,F)) \le \tau_n\left(\frac{|x| + |x-e_1| - 1}{2}\right).$$

Both ineq. reduce to equality if $x = se_1$ and $s \in (-\infty, 0) \cup (1, \infty)$.

Pf. Let $h \in \mathcal{GM}(\overline{\mathbb{R}}^n)$ with $(x, 0, e_1, \infty) \mapsto (-e_1, -y, y, e_1)$ where $|y|<1$, see Lemma 6.13. With $E_1 = [-|y|e_1, |y|e_1]$, $E' = [-y, y]$ and $F' = [-e_1, \infty] \cup [e_1, \infty]$ we have by Lemma 6.16

$$M(\Delta(E', F')) \le M(\Delta(E_1, F')) = \tau_n\left(\frac{(1-|y|)^2}{4|y|}\right)$$

Next by invariance of abs. ratios

$|x, \infty, 0, e_1| = |-e_1, e_1, -y, y|$ and $|x, \infty, e_1, 0| = |-e_1, e_1, y, -y|$

which give
$$|x| = \frac{|y-e_1|^2}{4|y|} \quad \text{and} \quad |x-e_1| = \frac{|y+e_1|^2}{4|y|}.$$

Thus
$$|x| + |x-e_1| - 1 = \frac{(1-|y|)^2}{2|y|}.$$

Setting $E = h^{-1}(E')$, $F = h^{-1}(F')$ we have

$$p(x) \le M(\Delta(E, F)) = M(\Delta(E', F')) \le \tau_n\left(\frac{|x|+|x-e_1|-1}{2}\right).$$

6.18. Cor. For $x \in \mathbb{R}^n \setminus \overline{B}^n$
$$\tau_n(|x-e_1|) \le p(x) \le \tau_n(|x-e_1|/2) \le \sqrt{2}\,\tau_n(|x-e_1|).$$

Pf. The first ineq. follows from 6.12 and the second one from 6.17. The third one follows from $\tau(ct)/\tau(t) \in [1, 1/\sqrt{c}]$ for $t > 0$ and $c \in (0,1)$ (AVVB, 11.25(1)).

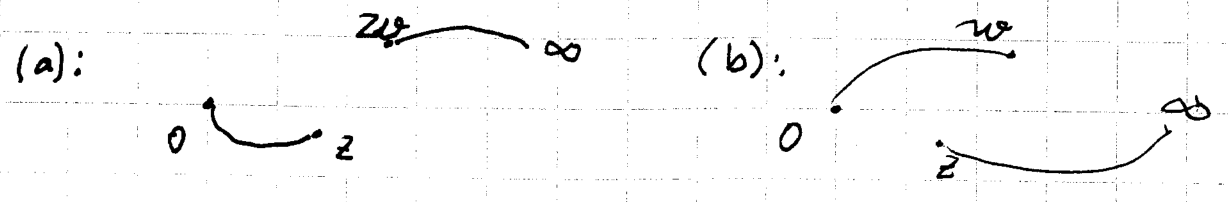
6.19. Cor. Let $G = \mathbb{R}^n \setminus \{0\}$ and $x, y \in G$ with $x \neq y$. Then
$$\lambda_G(x,y) \le \tau_n\left(\frac{|x-y| + ||x|-|y||}{2\min\{|x|,|y|\}}\right) \le \tau_n\left(\frac{|x-y|}{2\min\{|x|,|y|\}}\right)$$

Pf. By invar. under homotheties we may ass. $y = e_1$ and $|x| \ge 1$. Since $\min\{|x|,|y|\} = 1$ the result follows from 6.17.

6.20. <u>Thm</u> Let $G = \mathbb{C} \setminus \{0\}$ and $z, w \in G$, $z \neq w$. Then

$$\lambda_G(z,w) = \min\{p(z/w),\ p(w/z)\}.$$

<u>Proof.</u> In view of the def. of λ_G we have two possible choices of continua (a) and (b):



The choice (a) leads to $p(w/z)$ whereas (b) leads to $p(z/w)$.

6.21. <u>Thm</u> Let $G = \mathbb{R}^n \setminus \{0\}$ and $x, y \in G$, $x \neq y$, and let r_z be the similarity mapping with $r_z(0) = 0$ and $r_z(z) = e_1$. Then

$$\lambda_G(x,y) = \min\{p(r_x(y)),\ p(r_y(x))\}$$

<u>Pf.</u> We see that

$$|r_x(y) - e_1| = |x-y|/|x| \tag{6.22}$$

and that $r_z(x)$ takes the role of x/z.

6.22. <u>Thm</u> Let $G \subsetneq \mathbb{R}^n$ be a domain, $x, y \in G$, $x \neq y$, and $m(x,y) = \min\{d(x), d(y)\}$. Then

$$\lambda_G(x,y) \le \inf_{z \in \partial G} \lambda_{\mathbb{R}^n \setminus \{z\}}(x,y) \le \tau_n\left(\frac{|x-y|}{2m(x,y)}\right) \le \sqrt{2}\,\tau\left(\frac{|x-y|}{m(x,y)}\right)$$

<u>Pf.</u> The pf follows from 6.6, 6.18, 6.19.

6.23. Rmk. We now show that 6.22 may be a very crude upper bound for Jordan domains G. For $t \in (0, 1/5)$ let $G_t = B^n(-e_1, 1) \cup B^n(e_1, 1) \cup B^n(t)$. Then G_t is a Jordan domain and $6.22 \Rightarrow$

$$(*) \qquad \lambda_{G_t}(-e_1, e_1) \le \sqrt{2}\, \tau(2) \qquad \forall t$$

but

$$\lambda_{G_t}(-e_1, e_1) \le M(\Delta([-2e_1, -e_1], [e_1, 2e_1]; G_t)) \le \omega_{n-1} \left(\log \frac{1}{t}\right)^{1-n}$$

Because this tends to 0, $t \to 0$, we see that $(*)$ is very crude.

6.24. Rmk. Thm 6.17 and its corollaries 6.18 and 6.19 substitute the estimates on pages 106–111 of CGQM.

6.25. QED-domains. Let $E \subset \overline{\mathbb{R}}^n$ be closed. We say that E is a c-QED-set, $c \in (0, 1]$ if for every pair F_1, F_2 of continua $F_1, F_2 \subset \overline{\mathbb{R}}^n \setminus E$ we have

$$(6.26) \qquad M(\Delta(F_1, F_2; \overline{\mathbb{R}}^n \setminus E)) \ge c\, M(\Delta(F_1, F_2))$$

If $G \subset \overline{\mathbb{R}}^n$ is a domain and $\overline{\mathbb{R}}^n \setminus G$ is a c-QED set then we say that G is a c-QED domain ([GM])

6.27. Ex. (1) B^n is a $\frac{1}{2}$-QED domain

(2) If $\operatorname{cap} E = 0$ then E is a 1-QED set

(3) $B^2 \setminus [0, e_1)$ is not a c-QED set for any $c > 0$

(4) NED-set [V3] = 1-QED set [GM]

6.28. Thm. If G is a c-QED domain in $\mathbb{R}^n$, then

$$\lambda_G(x, y) \ge c\, \tau(4s^2 + 4s) \qquad \forall x, y \in G, \; x \ne y$$

where $s = |x-y| / \min\{d(x), d(y)\}$.