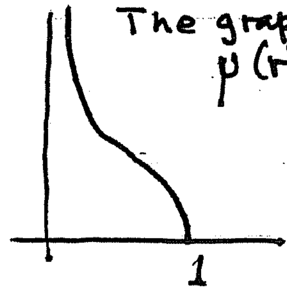
By (3.24) we may rewrite (3.23) as follows

$$(3.25) \qquad M(\Gamma) = \frac{4}{\pi}\,\mu\left(\frac{1-r}{1+r}\right)$$

We record the following inequality from LV2, pp. 61-62, $0<r<1$

$$(3.26) \qquad \log\frac{1}{r} < \log\frac{1+3\sqrt{1-r^2}}{r} < \mu(r) < \log\frac{2(1+\sqrt{1-r^2})}{r} < \log\frac{4}{r}\ .$$

The graph of $\mu(r)$




3.27. <u>Lemma</u> Let $\{\Gamma_j\}$ be separated curve families in $\overline{\mathbb{R}}^n$ with $\Gamma_j < \Gamma$ for all $j$. If $p>1$ then

$$M_p(\Gamma)^{1/(1-p)} \geq \sum_{j=1}^{\infty} M_p(\Gamma_j)^{1/(1-p)}\ .$$

<u>Proof.</u> [CGQM, 5.24]

3.28. <u>Rmk.</u> Let $\Gamma_{jk} = \Delta(S^{n-1}(j), S^{n-1}(k); B^n(j)\setminus \bar{B}^n(k))$. Then $\Gamma_{21} < \Gamma_{41}$, $\Gamma_{42} < \Gamma_{41}$ and $\Gamma_{21}, \Gamma_{42}$ are separate. By 3.12

$$M(\Gamma_{jk}) = \omega_{n-1}\left(\log\frac{j}{k}\right)^{1-n}$$

and $[a = \omega_{n-1}{}^{1/(1-n)}]$

$$M(\Gamma_{41})^{1/(1-n)} = a\log 4 = a\left(\log\frac{2}{1} + \log\frac{4}{2}\right) = a\log 4$$

with $a = \omega_{n-1}^{1/(1-n)}$, i.e. equality holds in 3.27.

3.29. <u>Lemma</u> Let $s\in(0,1)$ and

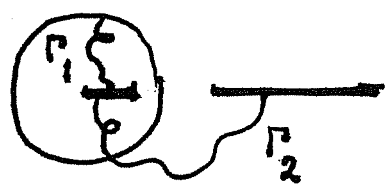
$$\Gamma_1 = \Delta([0,se_1], S^{n-1}; B^n),\ \Gamma_2 = \Delta([0,se_1],[e_1/s,\infty); \mathbb{R}^n)\ .$$

Then $M_p(\Gamma_1) = 2^{p-1}M_p(\Gamma_2)$ for $p>1$.

<u>Proof.</u> (Idea) Choose $\varrho\in F(\Gamma_2)$ s.t. $M_p(\Gamma_2) = \int\varrho^p\,dm$

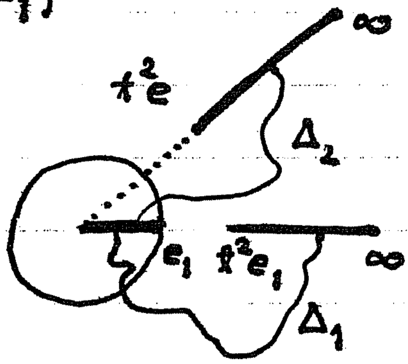
Symm. $\Rightarrow \varrho$ symm. w.r.t. $S^{n-1} \Rightarrow 2\varrho\,\chi_{B^n}\in F(\Gamma_1)$

$$\Rightarrow M_p(\Gamma_1) \leq 2^p\int_{B^n}\varrho^p\,dm = \underbrace{2^{p-1}\int_{\mathbb{R}^n}\varrho^p\,dm}_{\text{symm.}} = 2^{p-1}M_p(\Gamma_2)$$



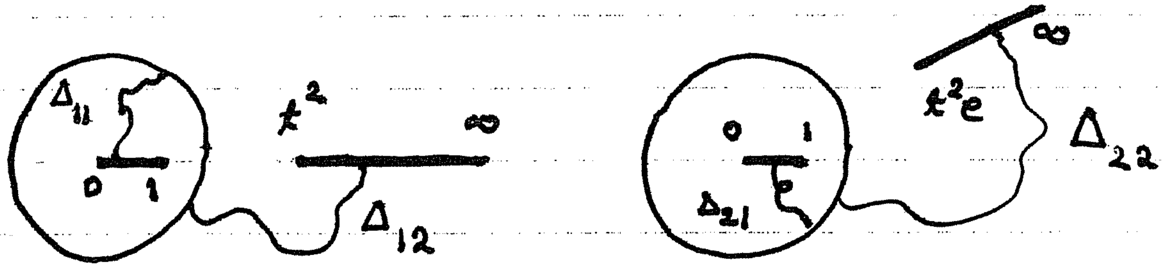

The proof for $\geq$ is similar.

3.30. Lause Let $\Delta_1 = \Delta([0,e_1],[t^2 e_1,\infty))$, $\Delta_2 = \Delta([0,e_1],[t^2 e,\infty))$ where $e \in S^{n-1}$ and $t>1$. Then $M(\Delta_2) \le M(\Delta_1)$




Proof. Let

$\Delta_{11} = \Delta([0,e_1], S^{n-1}(t))$, $\Delta_{12} = \Delta(S^{n-1}(t), [t^2 e_1,\infty))$
$\Delta_{21} = \Delta([0,e_1], S^{n-1}(t))$, $\Delta_{22} = \Delta(S^{n-1}(t), [t^2 e,\infty))$




Clearly $M(\Delta_{11}) = M(\Delta_{21})$, $M(\Delta_{12}) = M(\Delta_{22})$.
Let $f$ be an inversion in $S^{n-1}(t)$. Because $\Delta_{12} = f\Delta_{11}$, 3.19 gives

$$M(\Delta_{11}) = M(f\Delta_{11}) = M(\Delta_{12}).$$

Lemma 3.27 gives

$$M(\Delta_2)^{1/(1-n)} \ge M(\Delta_{21})^{1/(1-n)} + M(\Delta_{22})^{1/(1-n)} = 2M(\Delta_{11})^{1/(1-n)}$$

whereas by symmetry of $\Delta_1$ Lemma 3.29 gives

$$M(\Delta_{11}) = 2^{n-1} M(\Delta_1).$$

The desired ineq. follows easily from these results.

3.31. Thm. $M_p(\Gamma) = 0 \iff \exists\, \rho \in F(\Gamma) \cap L^p(\mathbb{R}^n)$ s.t. [Fuglede]

$$\int_\gamma \rho\, ds = \infty \quad \forall \text{ loc. rectifiable } \gamma \in \Gamma.$$

Proof. If $\rho$ satisfies these hypotheses then $\rho/k \in F(\Gamma)$ for $k=1,2,\ldots$ and

$$M_p(\Gamma) \le k^{-p} \int_{\mathbb{R}^n} \rho^p\, dm \to 0, \quad k \to \infty.$$

Therefore $M_p(\Gamma) = 0$. Conversely suppose that $M_p(\Gamma) = 0$ and choose $\rho_k \in F(\Gamma)$ s.t. $\int \rho_k^p \, dm < 4^{-k}$, $k = 1, 2, \ldots$ Write

$$\rho(x) = \left\{ \sum_{k=1}^{\infty} 2^k \rho_k(x)^p \right\}^{1/p}$$

whence $\rho \in L^p(\mathbb{R}^n)$. For every loc. rect. $\gamma \in \Gamma$ we have

$$\int_\gamma \rho \, ds \geq \int_\gamma 2^{k/p} \rho_k \, ds \geq 2^{k/p} \qquad k = 1, 2, \ldots$$

implying $\int_\gamma \rho \, ds = \infty$ for $\forall$ loc. rect. $\gamma \in \Gamma$.

3.32. Rmk. One can use Thm 3.31 to deduce Thm 3.6. Sometimes a family $\Gamma$ with $M_p(\Gamma) = 0$ is called *p-exceptional*

The family of all non–constant curves passing through a fixed point is $n$–exceptional as was pointed out in the paragraph following (5.15). One can show that such a family is not $p$–exceptional if $p > n$ (see [GOR, Chapter 3], [MAZ2]). We shall require this result in the following form, which is sometimes called the *spherical cap inequality.* For this result we introduce first an extension of the definition ($\overset{3}{\cancel{5}}$.1) of the $p$–modulus. Suppose that $S$ is a euclidean sphere in $\mathbf{R}^n$ with radius $r$ and $\Gamma$ is a family of curves in $S$. We equip $S$ with the restriction of the euclidean metric of $\mathbf{R}^n$ to $S$ and with the $(n-1)$–dimensional Hausdorff measure $m_{n-1}$ with $m_{n-1}(S) = \omega_{n-1} r^{n-1}$. Let $\mathcal{A}(\Gamma)$ be the set of all non–negative Borel–measurable functions $\rho \colon S \to \mathbf{R} \cup \{\infty\}$ with

$$\int_\gamma \rho \, ds \geq 1$$

for all locally rectifiable (with respect to the metric $ds$) curves $\gamma$ in $\Gamma$ and set

$$\mathsf{M}_n^S(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_S \rho^n \, dm_{n-1} \, .$$

For $\varphi \in (0, \pi)$ let $C(\varphi) = \{ z \in \mathbf{R}^n : z \cdot e_n \geq |z| \cos\varphi \}$.

3.33
~~5.28.~~ **Lemma.** *Let* $S = S^{n-1}(r)$, $\varphi \in (0,\pi]$, *let* $K$ *be the spherical cap* $S \cap C(\varphi)$, *and let* $E$ *and* $F$ *be non-empty subsets of* $K$.

(1) *Then*

$$\mathsf{M}_n^S(\Delta(E,F;K)) \geq \frac{b_n}{r}$$

*where* $b_n$ *is a positive number depending only on* $n$.

(2) *If* $K = S$, *i.e.* $\varphi = \pi$, *then* $b_n$ *may be replaced by* $c_n = 2^n b_n$ *in the above inequality.*

The proof of ~~5.28~~ 3.33 (see [V7, 10.9]) is based on an application of Hölder's inequality and Fubini's theorem. A similar method yields also the following improved form of ~~5.28~~ 3.33 ([R12, p. 57, Lemma 3.1], [GV1, p. 20, Lemma 3.8]).

3.34
~~5.29.~~ **Lemma.** *Assume that* $E$, $F$, *and* $K$ *are as in* ~~5.28~~ 3.33(1). *If* $\varphi \in (0, \frac{1}{2}\pi)$, *then*

$$\mathsf{M}_n^S(\Delta(E,F;K)) \geq \frac{d_n}{\varphi r}$$

*where* $d_n$ *depends only on* $n$.

3.35
~~5.30.~~ **Remark.** Throughout the book we will denote by $c_n$ the number in ~~5.28~~ 3.33(2). The number $b_n = 2^{-n} c_n$ has the following expression

$$\left\{\begin{aligned} &b_n = 2^{1-2n}\,\omega_{n-2}\, I_n^{1-n}, \quad b_2 = \frac{1}{2\pi}, \\ &I_n = \int_0^{\pi/2} \sin^{\frac{2-n}{n-1}} t \, dt\,. \end{aligned}\right. \tag{~~5.31~~ 3.36}$$

Because $\frac{2}{\pi} t \leq \sin t \leq t$ for $0 \leq t \leq \frac{1}{2}\pi$, it follows from (~~5.31~~ 3.36) that

$$(n-1)\left(\frac{\pi}{2}\right)^{1/(n-1)} \leq I_n \leq (n-1)\frac{\pi}{2}$$

for $n \geq 2$. One can show that $2^n c_n \to 0$ when $n \to \infty$ [AVV3].

By (~~5.1~~ 3.1), any admissible function $\rho$ yields an upper bound for $\mathsf{M}_p(\Gamma)$, that is $\mathsf{M}_p(\Gamma) \leq \int_{\mathbf{R}^n} \rho^p \, dm$. The problem of finding lower bounds for $\mathsf{M}_p(\Gamma)$ is much more difficult because then we need a lower bound for $\int_{\mathbf{R}^n} \rho^p \, dm$ for every admissible $\rho$. The next important lower bound for the modulus follows by integration from ~~5.28~~ 3.33 and ~~5.29~~ 3.34.

~~5.32~~ 3.37. **Lemma.** *Let* $0 < a < b$ *and let* $E$, $F$ *be sets in* $\mathbf{R}^n$ *with*

$$E \cap S^{n-1}(t) \neq \emptyset \neq F \cap S^{n-1}(t)$$

*for* $t \in (a,b)$. *Then*

$$\mathsf{M}(\Delta(E,F;B^n(b) \setminus B^n(a))) \geq c_n \log \frac{b}{a} .$$

*Equality holds if* $E = (ae_1, be_1)$, $F = (-be_1, -ae_1)$.

3.38 ~~5.33~~. **Corollary.** *If* $E$ *and* $F$ *are non–degenerate continua with* $0 \in E \cap F$ *then* $\mathsf{M}(\Delta(E,F)) = \infty$.

**Proof.** Apply 3.37 ~~5.32~~ with a fixed $b$ such that $S^{n-1}(b) \cap E \neq \emptyset \neq S^{n-1}(b) \cap F$ and let $a \to 0$. □

We next give a typical application of Lemma 3.37 ~~5.32~~. Unlike 3.37 ~~5.32~~ this application fails to give a sharp bound, but it yields adequate bounds in many cases (see e.g. Section 6). A sharp version of 3.39 ~~5.34~~. requires some information about spherical symmetrization, ~~will be given in Section 7 (see 7.32 and 7.33).~~

3.39 ~~5.34~~. **Lemma.** *Let* $t > r > 0$ *and let* $E \subset B^n(r)$ *be a connected set containing at least two points. Then*

$$\mathsf{M}(\Delta(S^{n-1}(t), E)) \geq c_n \log \frac{2t + d(E)}{2t - d(E)} .$$

**Proof.** Fix $u, v \in \overline{E}$ with $|u - v| = d(E) = d$ and choose $h \in \mathcal{GM}(B^n(t))$ with $h(u) = -se_1 = -h(v)$. By ~~(2.27)~~ (2.19)

$$d(E) = |u - v| \leq 2 \operatorname{th} \tfrac{1}{4}\rho(u,v) = 2 \operatorname{th} \tfrac{1}{4}\rho(h(u), h(v)) = 2s ,$$

where $\rho$ refers to the hyperbolic metric of $B^n(t)$. Applying 3.37 ~~5.32~~ to the annulus $B^n(te_1, t+s) \setminus \overline{B}^n(te_1, t-s)$ with $E = hE$ and $F = S^{n-1}(t)$ we obtain

$$\mathsf{M}(\Delta(S^{n-1}(t), E)) = \mathsf{M}(\Delta(S^{n-1}(t), hE)) \geq c_n \log \frac{t+s}{t-s}$$

$$\geq c_n \log \frac{2t + d(E)}{2t - d(E)} . \quad \square$$

(52)

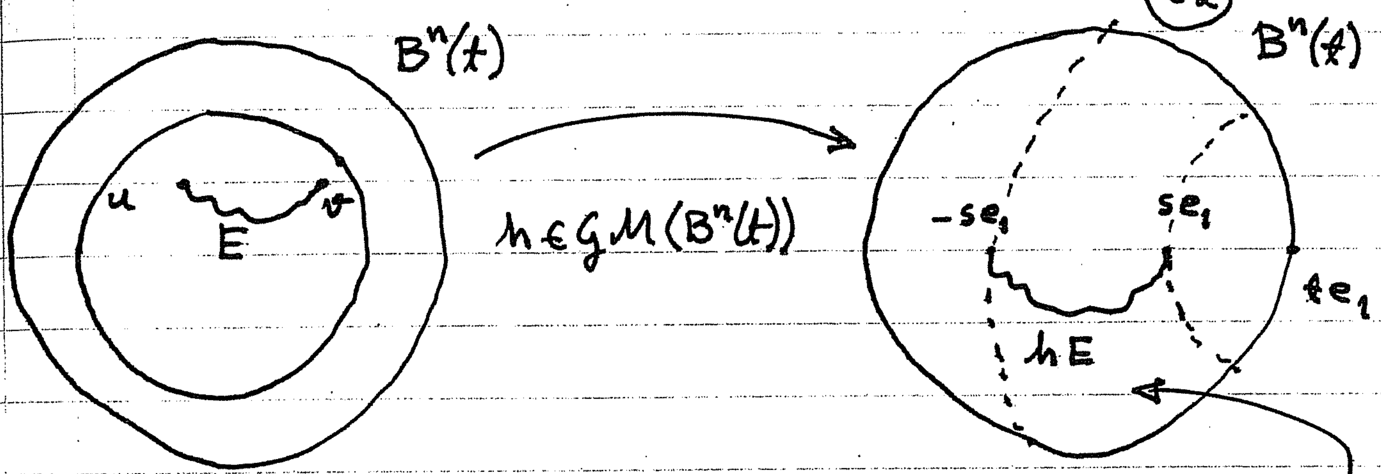



Figure for the pf of 3.39

The cap-ineq. applied in this annulus

We shall frequently apply the following lemma when proving lower bounds for the moduli of curve families. This lemma will be called the *comparison principle* for the modulus. In the applications of this lemma, the sets $F_3$ and $F_4$ will often be chosen to be non–degenerate continua (that is continua containing at least two distinct points) while the sets $F_1$ and $F_2$ will usually be very "small" sets when compared to $F_3$ and $F_4$.

**~~5.35~~ 3.40. Lemma.** *Let $G$ be a domain in $\overline{\mathbf{R}}^n$, let $F_j \subset G$, $j = 1,2,3,4$, and let $\Gamma_{ij} = \Delta(F_i, F_j; G)$, $1 \le i,j \le 4$. Then*

$$\mathsf{M}(\Gamma_{12}) \ge 3^{-n} \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24}),\ \inf \mathsf{M}(\Delta(|\gamma_{13}|, |\gamma_{24}|; G)) \},$$

*where the infimum is taken over all rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$.*

**Proof.** By ~~5.2~~ 3.2(1) we may assume that $F_j \neq \emptyset$, $j = 1,2,3,4$. Fix $\rho \in F(\Gamma_{12})$. If

$$\int_{\gamma_{13}} \rho \, ds \ge \tfrac{1}{3} \qquad \text{(~~5.36~~ 3.41)}$$

for every rectifiable $\gamma_{13} \in \Gamma_{13}$ or

$$\int_{\gamma_{24}} \rho \, ds \ge \tfrac{1}{3} \qquad \text{(~~5.37~~ 3.42)}$$

for every rectifiable $\gamma_{24} \in \Gamma_{24}$, then it follows from 5.8 and (5.1) that

(~~5.38~~) 3.43
$$\int_{\mathbf{R}^n} \rho^n \, dm \geq 3^{-n} \min\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24})\,\}\,.$$

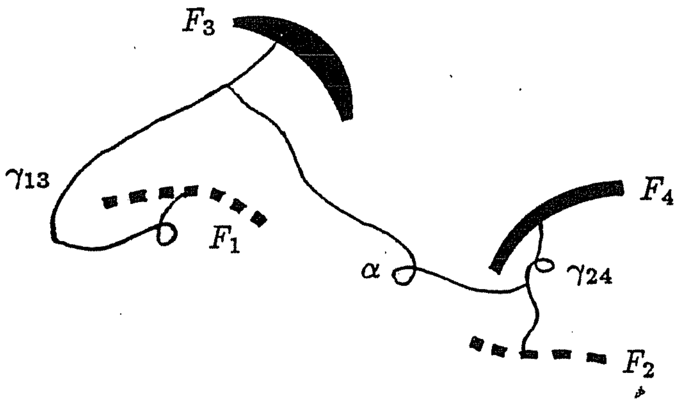



*Diagram 5.4.*

If both (~~5.36~~ 3.41) and (~~5.37~~ 3.42) fail to hold we select rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$. Because $\rho \in \mathcal{F}(\Gamma_{12})$ it follows that

$$\int_{\gamma_{13} \cup \alpha \cup \gamma_{24}} \rho \, ds \geq 1$$

for every locally rectifiable $\alpha \in \Delta = \Delta(|\gamma_{13}|, |\gamma_{24}|; G)$. Because both (~~5.36~~ 3.41) and (~~5.37~~ 3.42) fail to hold it follows from the last inequality that

$$\int_{\alpha} \rho \, ds \geq \tfrac{1}{3}$$

for each locally rectifiable $\alpha \in \Delta$. Hence

(~~5.39~~) 3.44
$$\int_{\mathbf{R}^n} \rho^n \, dm \geq 3^{-n} \mathsf{M}(\Delta) \geq 3^{-n} \inf \mathsf{M}\big(\Delta(|\gamma_{13}|, |\gamma_{24}|; G)\big)$$

where the infimum is taken over all rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$. In every case either (~~5.38~~ 3.43) or (~~5.39~~ 3.44) holds, and the desired inequality follows. □

~~5.40~~ 3.45. **Corollary.** *Let $F_j \subset \overline{\mathbf{R}}^n$ and $\Gamma_{ij} = \Delta(F_i, F_j)$, $1 \leq i, j \leq 4$. Then*

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24}),\ \delta_n(r)\,\}$$

*where $r = \min\{\, q(F_1, F_3),\ q(F_2, F_4)\,\}$ and*

$$\delta_n(r) = \inf \mathsf{M}\big(\Delta(E, F)\big)\,.$$

*Here the infimum is taken over all continua $E$, $F$ in $\overline{\mathbf{R}}^n$ such that $q(E) \geq r$, $q(F) \geq r$.*

It is clear that $\delta_n(0) = 0$ in 5.40. In fact, this follows from 5.18(2) if we choose $r \in (0, 1/\sqrt{2})$, set $s = \sqrt{1-r^2}$, and let $r \to 0$. We are going to show that $\delta_n(r) > 0$ for $r > 0$. To this end the following corollary will be needed.

3.46 ~~5.41~~. **Corollary.** *If* $x \in \mathbf{R}^n$, $0 < a < b < \infty$, *and* $F_1, F_2 \subset B^n(x,a)$, $F_3 \subset \mathbf{R}^n \setminus B^n(x,b)$, $\Gamma_{ij} = \Delta(F_i, F_j)$, *then*

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\Big\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}),\ c_n \log \frac{b}{a} \Big\}, \tag{1}$$

$$\mathsf{M}(\Gamma_{12}) \geq d(n, b/a) \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \}. \tag{2}$$

**Proof.** We apply the comparison principle 3.40 ~~5.35~~ with $G = \mathbf{R}^n$ and $F_3 = F_4$ to get a lower bound for $\mathsf{M}(\Gamma_{12})$. It follows from 3.37 ~~5.32~~ that the infimum in the lower bound of 3.40 ~~5.35~~ is at least $c_n \log \frac{b}{a}$ and thus (1) follows. For the proof of (2) we observe that by 5.3 and (5.14)

$$\max\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \} \leq A = \omega_{n-1} \Big( \log \frac{b}{a} \Big)^{1-n}.$$

By part (1) we get

$$\begin{aligned} \mathsf{M}(\Gamma_{12}) &\geq 3^{-n} \min\Big\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}),\ \frac{1}{A}\Big( c_n \log \frac{b}{a} \Big) \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \} \Big\} \\ &\geq d(n, b/a) \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \} \end{aligned}$$

where $d(n, b/a) = 3^{-n} \min\{ 1,\ \frac{1}{A} c_n \log(b/a) \}$. □

3.47 ~~5.42~~. **Lemma.** *For* $n \geq 2$ *there are positive numbers* $d$ *and* $D$ *with the following properties.*

(1) *If* $E, F \subset B^n(s)$ *are connected and* $d(E) \geq st$, $d(F) \geq st$, *then* $\mathsf{M}(\Delta(E,F)) \geq dt$.

(2) *If* $E, F \subset \overline{\mathbf{R}}^n$ *are connected and* $q(E) \geq t$, $q(F) \geq t$, *then* $\mathsf{M}(\Delta(E,F)) \geq \delta_n(t) \geq Dt$.

**Proof.** (1) By 3.39 ~~5.34~~ we obtain

$$\mathsf{M}(\Delta(S^{n-1}(2s), E)) \geq c_n \log \frac{4s + ts}{4s - ts} \geq \tfrac{1}{2} c_n (\log 2) t$$

and similarly $\mathsf{M}(\Delta(S^{n-1}(2s), F)) \geq \frac{1}{2} c_n (\log 2) t$. Applying 3.46 ~~5.41~~(1) with $F_1 = F$, $F_2 = E$, and $F_3 = S^{n-1}(2s)$ and the above estimates we get

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{ \tfrac{1}{2} c_n (\log 2) t,\ c_n \log 2 \} \geq dt$$

where $d = \frac{1}{2} \cdot 3^{-n} c_n \log 2$.

(2) Observe first that both the first and last expressions in the asserted inequality remain invariant under spherical isometries ~~(see 5.17)~~. By performing a preliminary spherical isometry if necessary we may assume that $-re_1 \in E$, $re_1 \in F$, and $r \in [0,1]$ ~~(cf. 1.25(1))~~. Let $E_1$ $(F_1)$ be that component of $E \cap \overline{B}^n(2)$ (of $F \cap \overline{B}^n(2)$, resp.) which contains $-re_1$ ($re_1$). Then

$$d(E_1) \geq q(E_1) \geq \min\{t,\ q(S^{n-1}, S^{n-1}(2))\} \geq t/\sqrt{10},$$

and likewise $d(F_1) \geq t/\sqrt{10}$. The proof of (2) follows from (1) with $D = d/\sqrt{10}$.

□

For the case of connected sets $E, F$ the above results enable us to prove many useful inequalities for $M(\Delta(E,F))$. The next result applies also for disconnected sets $E, F$.

3.48. Thm For $n \geq 2$ there exist posit. numbers $d_1, \ldots, d_4$ and a set function $c(\cdot): \text{pot}(\overline{\mathbb{R}}^n) \to [0, \infty)$ s.t.

(1) $c(E) = c(hE)$ $\forall\, E \subset \overline{\mathbb{R}}^n$ $\forall$ $q$-isometry $h$

(2) $c(\emptyset) = 0$, $A \subset B \subset \overline{\mathbb{R}}^n \Rightarrow c(A) \leq c(B)$

$c\left(\bigcup_{k=1}^{\infty} E_k\right) \leq d_1 \sum_{k=1}^{\infty} c(E_k)$

(3) If $E \subset \overline{\mathbb{R}}^n$ is compact then $c(E) > 0 \Leftrightarrow \operatorname{cap} E > 0$. Also $c(\overline{\mathbb{R}}^n) \leq d_2 < \infty$.

(4) If $E$ is connected, then $c(E) \geq d_3\, q(E)$

(5) $M(\Delta(E,F)) \geq d_4 \min\{c(E), c(F)\}$ $\forall\, E, F \subset \overline{\mathbb{R}}^n$

(6) For $n \geq 2$, $t \in (0,1)$ there exists $d_5(n,t)$ s.t. $M(\Delta(E,F)) \leq d_5 \min\{c(E), c(F)\}$ $\forall\, E, F \subset \overline{\mathbb{R}}^n$ $q(E,F) \geq t$.

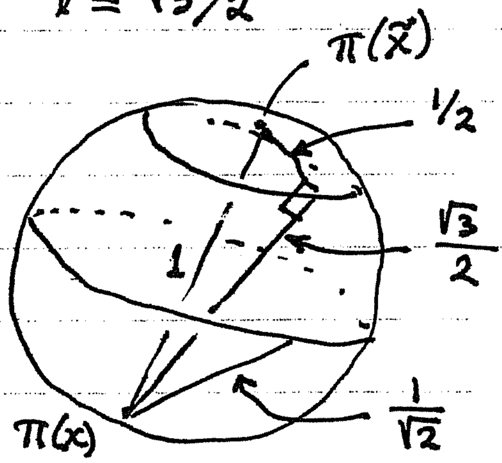
Note. The condition $\operatorname{cap} E > 0$ will be defined later.

3.49. <u>The construction of $c(E)$</u>. For $E \subset \overline{\mathbb{R}}^n$, $x \in \overline{\mathbb{R}}^n$, $0 < r < t < 1$, we write (recall $Q(z,r) = \{x \in \overline{\mathbb{R}}^n : q(z,x) < r\}$ )

$$(3.50)\quad \begin{cases} m_t(E,r,x) = M(\Delta(\partial Q(x,t), E \cap Q(x,r))) \\ m(E,x) = m_t(E, 1/\sqrt{2}, x), \quad t = \sqrt{3}/2 \end{cases}$$

Now define

$$(3.51)\quad \begin{cases} c(E,x) = \max\{m(E,x), m(E,\tilde{x})\} \\ c(E) = \inf\{c(E,x) : x \in \overline{\mathbb{R}}^n\} \end{cases}$$




We have

$$(3.52)\quad M(\Delta(\partial Q(z,t), \partial Q(z,s))) = M(\Delta(\partial Q(0,t), \partial Q(0,s))) = \omega_{n-1}\left(\log \frac{t}{r}\frac{\sqrt{1-r^2}}{\sqrt{1-t^2}}\right)^{1-n} < \omega_{n-1}\left(\log \frac{t}{r}\right)^{1-n}$$

and

$$(3.53)\quad m(E,x) \le m(\overline{\mathbb{R}}^n, x) = M(\Delta(\partial Q(0, \tfrac{\sqrt{3}}{2}), \partial Q(0, 1/\sqrt{2}))) = \omega_{n-1}(\log \sqrt{3})^{1-n}.$$

For the proof of Thm 3.48 see [CGQM, Section 6].