General remarks.

(1) The web-address
http://mathworld.wolfram.com
is useful for finding unfamiliar terms.

(2) For the course credit also various "computer proofs" of homework will be accepted.

I CONFORMAL GEOMETRY

Conformal geometry is study of geometry using basic notions of group theory.

Metric space (X, d)

Group of isometries $\Gamma = \{\gamma : X \to X \text{ bijection} : \gamma \text{ isometry}\}$

Equivalence classes of configurations: $\{x_1, \dots, x_p\} \sim \{y_1, \dots, y_p\}$ if $\exists\, \gamma \in \Gamma$ with $y_k = \gamma(x_k)$ for all k.

Ex. $(\mathbb{R}^n, |\,|)$. $\Gamma_1 = \{\text{translations}\}$, $\Gamma_2 = \{\text{rotations}\}$.

In what follows, Γ is a subgroup of Möbius transform.

1. Möbius transformations

Beardon (1983), Ahlfors (1981), Carathéodory vol. I

$e_1, \dots, e_n$ unit vectors, e.g. $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ ($= R^n$)

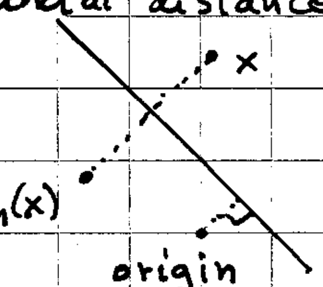
$\langle x, y \rangle = (x \mid y) = \sum_{j=1}^{n} x_j y_j = x \cdot y$; $|x| = (x \mid x)^{1/2}$.

$a \in \mathbb{R}^n \setminus \{0\}$: $[a, \infty] = \{ta : t \geq 1\} \cup \{\infty\}$, $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$

$t \in \mathbb{R}$, $a \in \mathbb{R}^n \setminus \{0\}$: $P(a, t) = \{x \in \mathbb{R}^n : (x \mid a) = t\} \cup \{\infty\}$

$P(a, t)$ is $(n-1)$-dim. plane in $\overline{\mathbb{R}}^n$, $\perp a$, and at distance $t/|a|$ from the origin.

1.1. Elementary transformations



(1) Reflection in the plane $P(a, t)$:

$$f_1(x) = x - 2\big((x \mid a) - t\big)\frac{a}{|a|^2}, \quad f_1(\infty) = \infty$$

(2) Inversion (reflection) in the sphere $S^{n-1}(a, r)$:

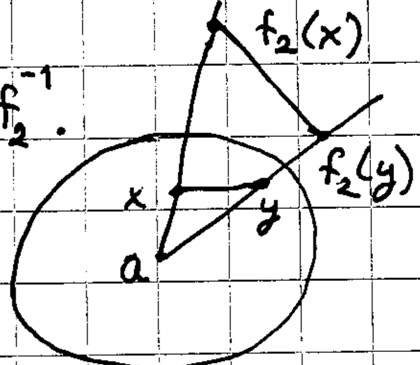
$$f_2(x) = a + \frac{r^2(x-a)}{|x-a|^2}, \quad f_2(a) = \infty, \quad f_2(\infty) = a$$

(3) Translation: $f_3(x) = x + a$, $a \in \mathbb{R}^n$, $f_3(\infty) = \infty$

(4) Stretching (dilation): $f_4(x) = kx$, $f_4(\infty) = \infty$, $k > 0$

(5) Orthogonal mapping: f_5 is a linear map and $|f_5(x)| = |x|$ for all $x \in \mathbb{R}^n$, $f_5(\infty) = \infty$ (rotation)

1.2. Rmk. The translation $x \mapsto x+a$ is a composition of two reflections, one in $P(a,0)$ and the other in $P(a, |a|^2/2)$. The stretching $x \mapsto kx$ is composit. of inversions in $S^{n-1}(0,1)$ and $S^{n-1}(0,\sqrt{k})$. It can be shown [Beardon, s.23, Thm 3.1.3] that an orthogonal map can be composed of at most $(n+1)$ reflections.



1.3. Rmk. If f_2 is as in 1.1(2), then $f_2 = f_2^{-1}$.

Similar triangles $\Rightarrow$

(1.4) $$|f_2(x) - f_2(y)| = \frac{r^2 |x-y|}{|x-a|\,|y-a|} .$$

1.5. Def. A homeom. $f: \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is called a Möbius transformation, if there exist an integer $p \geq 1$ and elem. transformations $g_1, \ldots, g_p$ such that $f = g_1 \circ \cdots \circ g_p$. (By 1.2 one can require, furthermore, that each g_j is an inversion or reflection.)

It can be shown that the set of all Möbius transf. of $\overline{\mathbb{R}}^n$ constitutes a group $GM(\overline{\mathbb{R}}^n)$, with the composition of maps as the group operation. Abbr. $GM = GM(\overline{\mathbb{R}}^n)$ and

$$GM(D) = \{f \in GM : fD = D\}, \quad D \subset \overline{\mathbb{R}}^n .$$

For example, the set of all ortog. maps $O(n)$ is a subgroup of GM. A map $f \in GM$ is called a similarity, if there exists $c > 0$ such that $|f(x) - f(y)| = c|x-y|$ for all x, y.

1.6. Def. Let $D, D' \subset \overline{\mathbb{R}}^n$ be domains. A C^1-homeom. $f: D \to D'$ is called orientation preserving (or sense pres.) [op] if $J_f(x) > 0$ for all $x \in D \setminus \{\infty, f^{-1}(\infty)\}$. If $J_f(x) < 0$ for all x then f is orientation reversing (or sense reversing) [or].

A reflection in a plane or sphere is or. Facts:
$op \circ or = or = or \circ op$, $or^{-1} = or$, $or \circ or = op$, $op^{-1} = op$.
Write $M(\overline{\mathbb{R}}^n) = M = \{f \in GM : f \text{ is op}\}$
$$M(D) = \{f \in M : fD = D\},\quad D \subset \overline{\mathbb{R}}^n.$$
For a map $f \notin C^1$ op and or will be defined later.

1.7. Rmk. Every $f \in GM(\overline{\mathbb{R}}^n)$ can be extended to a map $\tilde{\tilde{f}} \in GM(\overline{\mathbb{R}}^{n+1})$ s.t. $\tilde{\tilde{f}}|\overline{\mathbb{R}}^n = f$ (Poincaré's extension). Here $\mathbb{R}^n = \{(x_1, \dots, x_n, 0) \in \mathbb{R}^{n+1}\}$. Note that it is enough to do this for reflections.

1.8. Stereographic projection. Ster. projection is the map $\pi : \overline{\mathbb{R}}^n \to S^n(e_{n+1}/2, 1/2)$ defined by

$$(1.9)\quad \pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2};\ x \in \overline{\mathbb{R}}^n,\ \pi(\infty) = e_{n+1}.$$

Thus π is the reflection in $S^n(e_{n+1}, 1) \subset \mathbb{R}^{n+1}$ restricted to $\overline{\mathbb{R}}^n$. We will identify ster. proj. with this reflection. Because $g^{-1} = g$ for all reflections g, we see that π maps $S^n(e_{n+1}/2, 1/2)$ onto $\overline{\mathbb{R}}^n$.

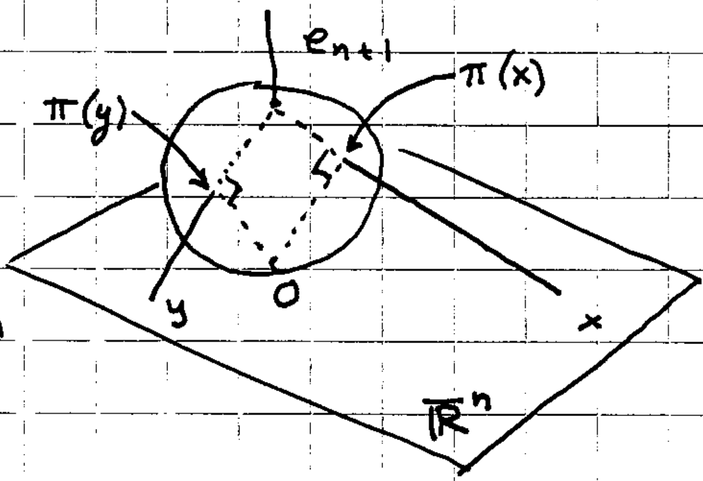


Figure
The Riemann sphere $S^n(e_{n+1}/2, 1/2)$ and the stereographic projection

The q metric on $\overline{\mathbb{R}}^n$ is defined by

(1.10) $q(x,y) = |\pi(x) - \pi(y)|\,;\ x,y \in \overline{\mathbb{R}}^n$

Sometimes, q is called the <u>spherical metric</u> (or the <u>chordal</u> metric). Because π is an inversion $\left.\begin{matrix}(1.4)\\(1.9)\end{matrix}\right\}\Rightarrow$

(1.11) $$\begin{cases} q(x,y) = \dfrac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}\,; & x \neq \infty \neq y, \\ q(x,\infty) = 1/\sqrt{1+|x|^2}\,; & x \neq \infty. \end{cases}$$

Given $x \in \overline{\mathbb{R}}^n$, its <u>antipodal point</u> is defined as foll.

(1.12) $\tilde{x} = -x/|x|^2,\ x \in \mathbb{R}^n \setminus \{0\},\ \tilde{0} = \infty,\ \tilde{\infty} = 0.$

By (1.11) we see that $q(x,\tilde{x}) = 1$ and therefore $\pi(x)$ and $\pi(\tilde{x})$ are diametrically opposite points on the Riemann sphere.

From (1.11) it follows immediately that

(1.13) $q(x,y) \le \min\{1, |x-y|\},\ x,y \in \mathbb{R}^n.$

1.14. <u>Exer.</u> (1) Show that $q(x,y) = q(x/|x|^2, y/|y|^2)$ and

$$q(x,y) \le |x-y| / \max\{|x||y|,\ 2\sqrt{|x||y|}\}, \quad x,y \neq 0,$$

$$\frac{|x-y|}{(1+|x|)(1+|y|)} \le q(x,y) \le \frac{2|x-y|}{(1+|x|)(1+|y|)}$$

Show also that $q(x,y) \ge |x-y|/2$ for $x,y \in B^n$.

1.15. <u>Rmk.</u> Some further inequalities can be found in [GQM and AVVB. For instance: $|x-y| \ge \dfrac{2q}{1+\sqrt{1-q^2}},\ q = q(x,y)$

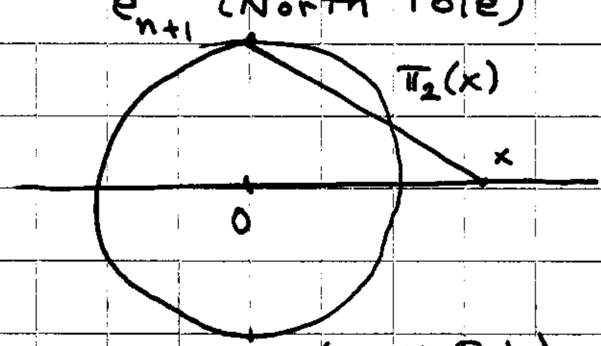
Recall that bilipschitz maps and isometries were defined in the introduction for general metric spaces. Here these will be considered for the q metric.

The inversion in S^{n-1} and orthogonal maps are isometries of the q metric whereas the translation $x \mapsto x + e_1$ and the stretching $x \mapsto x/2$ are not. Reflection in a plane and translation are euclidean isometries.

1.16. <u>Rmk.</u> The inversion in $S^n(e_{n+1}, \sqrt{2})$

$$\pi_2(x) = e_{n+1} + \frac{2(x - e_{n+1})}{|x - e_{n+1}|^2}\,;\ x \in \mathbb{R}^n,\ \pi_2(\infty) = e_{n+1}$$

is sometimes also called the stereogr. projection (but not in these notes)



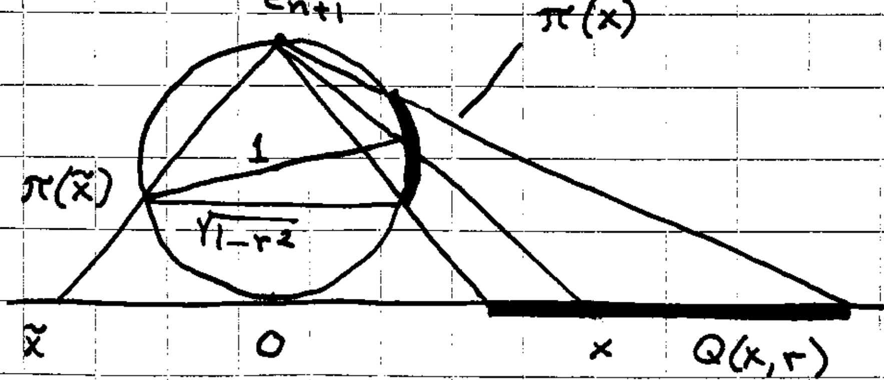
During this course $\overline{\mathbb{R}}^n$ topology of $\overline{\mathbb{R}}^n$ is indeed the topology defined by the metric q. In particular, the notion of convergence is defined in terms of q. Therefore the point ∞ has no exceptional role.

1.17. <u>The q-balls.</u> For $x \in \overline{\mathbb{R}}^n$ and $r \in (0,1)$ set

$$Q(x,r) = B_q(x,r) = \{z \in \overline{\mathbb{R}}^n : q(x,z) < r\}.$$

The Pythagorean theorem implies that

(1.18) $Q(x,r) = \overline{\mathbb{R}}^n \setminus \overline{Q}(\tilde{x}, \sqrt{1-r^2})$



Formula (1.18) shows, in particular, that $\pi Q(x, 1/\sqrt{2})$ is a hemisphere of the Riemann sphere $S^n(e_{n+1}/2, 1/2)$. If $r \in (0, 1/\sqrt{2}]$ then $Q(x,r)$ and $\partial Q(x,r)$ both have the same spherical diameters.

1.19. Exer. Show that $\mathbb{R}^n_+ = \mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\} = Q(e_n, \frac{1}{\sqrt{2}})$ and $Q(0,r) = B^n(r/\sqrt{1-r^2})$. Show that $q(\bar{B}^n(b, \sqrt{1+|b|^2})) = 1$.

1.20. Absolute ratio For an ordered quadruple $a,b,c,d \in \overline{\mathbb{R}}^n$ [distinct], the absolute ratio is defined by

$$|a,b,c,d| = \frac{q(a,c)\, q(b,d)}{q(a,b)\, q(c,d)} \tag{1.21}$$

1.22. Exer. For $a,b,c,d \in \mathbb{R}^n$ we have

$$|a,b,c,d| = \frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}. \quad \text{[Hint: (1.11)]}$$

The most important property of the absolute ratio is the following GM-invariance:

1.23. Thm [Beardon] If $f \in GM$ and $a,b,c,d \in \overline{\mathbb{R}}^n$, then $|f(a), f(b), f(c), f(d)| = |a,b,c,d|$.

1.24. Rmk. (1) The abs. ratio depends on the order of the points and permutation of points will give 6 possible values say $r_1, \ldots, r_6$ where $r_4 = 1/r_1$, $r_5 = 1/r_2$, $r_6 = 1/r_3$. E.g.

$|0, e_1, x, \infty| = |x| = 1/|0, x, e_1, \infty|$,

$|0, e_1, \infty, x| = |x - e_1| = 1/|0, \infty, e_1, x|$,

$|0, \infty, x, e_1| = |x|/|x-e_1| = 1/|0, x, \infty, e_1|$.