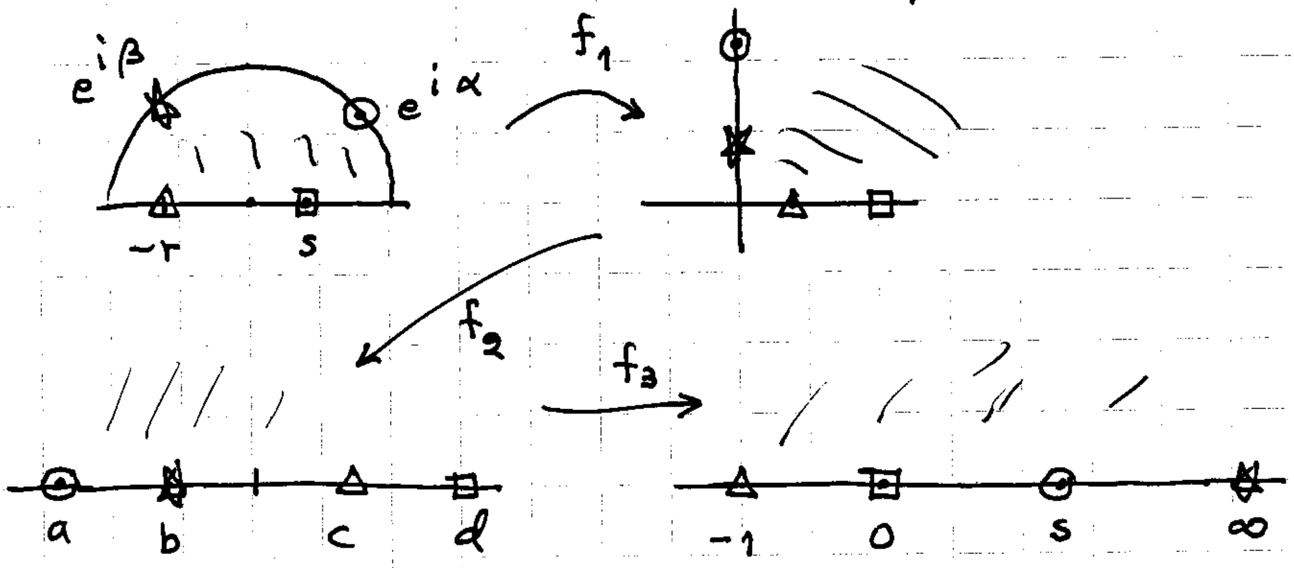
<u>Ex.</u> Let $0 < \alpha < \beta < \pi$, $r, s \in (0,1)$ and $D = H^2 \cap B^2$.
Find $M(D; -r, s, e^{i\alpha}, e^{i\beta})$.

<u>Solution</u> Map D onto H^2 (use simplified def.) conf.
We use several conformal maps



The main step is to find f_1 (complex variables course), it is $f_1(z) = \frac{1+z}{1-z}$. $f_2(z) = z^2$. The map f_3 is a Möbius transformation which can be found by the invariance of i.e.

$$(-1, 0, s, \infty) = \frac{-1-s}{-1} \cdot \frac{0-\infty}{s-\infty} = s+1$$

$$(c, d, a, b) = \frac{c-a}{c-d} \cdot \frac{d-b}{a-b}$$

these are equal

$\Rightarrow s = \dots$ Finally use the formulas for f_1 and f_2 to get a, b, c, d. [Homework!]

<u>Analytic function</u>: $\begin{cases} f: G \to \mathbb{C} \\ f(z) = u(x,y) + i v(x,y) \quad z = x + iy \end{cases}$

f anal $\iff$ Cauchy-Riemann $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\iff u, v$ harmonic : $\Delta u = 0$, $\Delta v = 0$; $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$f(z)$ anal. $\Rightarrow f^{(k)}(z)$ anal, $k = 1, 2, \ldots$

$f(z)$ anal, not constant $\Rightarrow f^{-1}(y)$ is $\begin{cases} \text{discrete set} \\ \text{isolated} \end{cases}$ i.e. has no accumulation points in G.

f conformal $\iff$ f anal and injective

f conf. : small circles $\longrightarrow$ circles

angles are preserved

$f: G \to \mathbb{C}$ non-const. anal. $\Rightarrow f$ is open

$\Rightarrow f \,\big|\, G \setminus \underbrace{\{z : f'(z) = 0\}}_{= Z}$ loc. conf.

The set Z has no accumulation points in G.

A domain is simply-connected (in $\mathbb{C}$) if it has no holes in it.

<u>Riemann's mapping theorem</u> A s.c. domain $G \subsetneq \mathbb{C}$ can be mapped conformally onto the unit disk B^2.

<u>Rmk</u> 1) The expression for this conf. map is, in general, not known

2) For a polygonal domain G, the Schwarz-Christoffel formula gives a semi-explicit formula

3) For polygonal G one can carry out the S-C map numerically with suitable software

<u>Modulus of continuity</u> Let $\omega : [0,\infty) \to [0,\infty)$, $\omega(0) = 0$ be a homeomorphism and let (X, d_1), (Y, d_2) be metric spaces. We say that $f : (X, d_1) \to (Y, d_2)$ is uniformly continuous if for all $x, y \in X$ (ω is the modulus of continuity)

$$d_2(f(x), f(y)) \le \omega(d_1(x,y)).$$

<u>Hölder, Lipschitz</u> We say that f is

there exist $c > 0$, $\alpha \in (0,1]$ such that for all $t > 0$

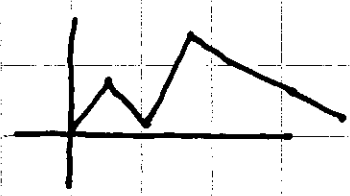
(1) Hölder-continuous, if $\omega(t) \le c\, t^{\alpha}$

(2) Lipschitz-cont. if f is Hölder with $\alpha = 1$

(3) bilipschitz, if f and f^{-1} are homeos and Lipschitz

(4) isometry if f is Lipschitz with $c = 1$.

<u>Construction of maps.</u> A basic method of construction is to define the mapping in a piecewise manner. For instance, a piecewise linear map could have a graph like this:



For the case $f : \mathbb{R}^n \to \mathbb{R}^n$, $n > 1$, we could make the construction coordinatewise. Sometimes it is handy to use polar or cylindrical coordinates.

(1) $f : \mathbb{R} \to \mathbb{R}$, $|f'(x)| \le L < \infty \Rightarrow |f(x) - f(y)| \le L\, |x - y| \ \forall x, y$
$\Rightarrow f : (\mathbb{R}, |\,|) \to (\mathbb{R}, |\,|)$ is L-Lipschitz

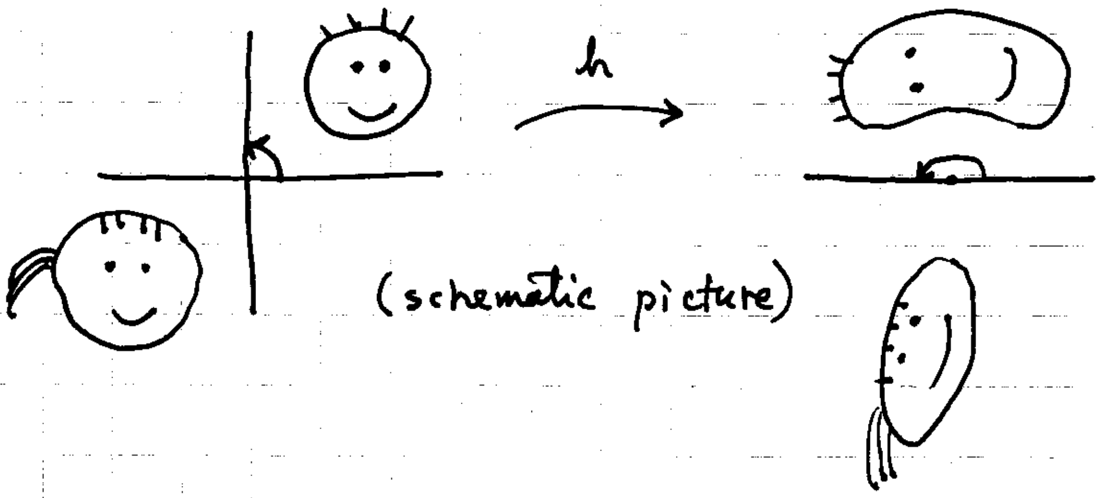
(2) $f : \mathbb{R} \to \mathbb{R}$, $1/M \le |f'(x)| \le M \Rightarrow f$ is M-bilipschitz

(3) $h : \mathbb{R}^2 \to \mathbb{R}^2$, $h(x,y) = (x, g(y))$, g Lip $\Rightarrow h$ is Lip

(4) (r, φ) polar coordin. in $\mathbb{R}^2$. Let $h : [0, 2\pi) \to [0, 2\pi)$ be defined

$$\begin{cases} h(\varphi) = 2\varphi, & 0 \le \varphi \le \pi/2, \\ h(\varphi) = \pi + c\,(\varphi - \pi/2) \end{cases}$$

where c is chosen so that $h(2\pi -) = 2\pi$.



(schematic picture)

(5) $(r, \varphi) \longmapsto (r^\alpha, \varphi)$; $0 < \alpha \leq 1$

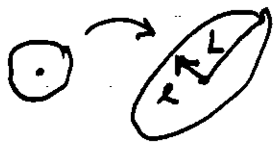
(6) $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = |x|^{\alpha - 1} x$. The radial mapping. This generalizes the map in (5) to $\mathbb{R}^n$.

<u>General picture</u> BILIP $\subsetneq$ QC $\subsetneq$ Hölder

	$n=2, K=1$	$n \geq 2$, $K \geq 1$
inj.	CONF	QC
general	ANAL	QR

$\exists\, K \geq 1$:

Locally: "circles $\longrightarrow$ ellipses" a.e



$\frac{L}{\ell} \leq K$

Globally: more difficult

However: quasicircles, i.e. sets $f(S^1)$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ qc need not be locally rectifiable