Introduction to Conformal Geometry and Quasiconformal Maps
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Exercise 12, 2011-04-19 `File: icg1112.tex, 2011-4-4,9.31`

1. Let $f : B^2 \to B^2 \setminus \{0\} \equiv G$ be the analytic function defined in h1001, $f(z) = \exp(g(z))$ when $g(z) = -(1+z)/(1-z), z \in B^2$. Estimate for $t \in (0,1)$

$$\sup\{k_G(f(0), f(z)) : |z| = t\}.$$

Hint: Consider $k_G(|f(0)|, |f(z)|)$.

2. Let $G \subset \mathbf{R}^n$ be a domain, $x_0 \in G, G_1 = G \setminus \{x_0\}, t \in (0, 1/2]$. Show that there is a constant $c \geq 1$ such that for all $x, y \in G \setminus B^n(x_0, td(x_0))$ $k_{G_1}(x, y) \leq ck_G(x, y)$.

3. Find a counterpart of the Schwarz lemma for
(a) $K-$qm mappings $f : Q(z, r) \to Q(w, s)$, $f(z) = w$.
(b) $K-$qr mappings $f : \mathbf{H} \to \mathbf{H}$, $f(e_n) = e_n$.

4. Let $f : \mathbf{B} \to \mathbf{B}$ be $K-$qr and $u(x) = 1 - |f(x)|$. Show that the Harnack inequality holds for u.

5. Let $G \subset \mathbf{R}^n$ be a domain $x, y, z \in G$ with $|x-y| = d(x)/2$ and $|x-z| > d(x)$. Find a lower bound for $\lambda_G(x, z)$ in terms of $\lambda_G(x, y)$ and $k_G(z, y)$. Hint: You may reduce the former case ($\lambda_G(x, z)$) to the latter case ($\lambda_G(x, x)$) by use of an auxiliary qc mapping as follows. It is well-known [GP] that for a domain $D \subset \mathbf{R}^n$ and $x, y \in D$ there is a $K-$quasiconformal mapping $f : D \to D$ with $f(z) = z$ for all $z \in \partial D$ with $f(x) = y$, $K \leq \exp(c_1 k_D(x, y))$ where $c_1 > 0$ is a constant.

6. Let $f : \mathbf{B} \to Z$, $Z = \{x \in \mathbf{R}^n : \Sigma_{j=1}^{n-1} x_j^2 < 1\}$ be $K-$qr, $f(0) = 0$. Show that

$$|f(x)| \leq AK(\log \frac{1 + |x|}{1 - |x|} + B),$$

where A, B depend only on n. [Hint: 5.29[CGQM] and μ-metric.]