

Introduction to Conformal Geometry and Quasiconformal Maps
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Winter 2011 / Vuorinen

Exercise 6, 2011-02-28 File: icg1106.tex, 2011-2-19,16.11

1. Show that for all $x, y \in G \subsetneq \mathbf{R}^n$ the inequality

$$\inf_{\gamma \in \Gamma_{xy}} \log \left(1 + \frac{\ell(\gamma)}{\min\{d(x), d(y)\}} \right) \leq k_G(x, y),$$

where Γ_{xy} is the family of all curves connecting the points x and y within G , and $d(x) = \text{dist}(x, \partial G)$. Hint. Recall the normal representation of a curve.

2. Let $D = \mathbf{R}^2 \setminus \{te_1 : t \geq 0\}$. Show that there is no constant $C > 0$ such that

$$k_D(x, y) \leq C j_D(x, y), \forall x, y \in D.$$

3. Let us define

$$m(x, y) = |x - y|^{1/2}$$

for $x, y \in \mathbf{R}$. Show that m is a metric and $[0, 1]$ is not rectifiable with respect to m . (Hint: length of $[0, 1]$ is

$$\sup \left\{ \sum_{i=1}^n m(x_i, x_{i+1}) : n \in \mathbb{N}, x_1 = 0, x_{n+1} = 1, x_k < x_{k+1} \right\}.$$

4. Let $r_k \in (0, 2^{-k-3})$, let $B_k = B^n(2^{-k}e_n, r_k)$, and let $E = \cup B_k$.

(a) Show that the numbers r_k can be chosen so that $M(\Delta(E, \partial \mathbf{H})) = \infty$.

(b) Show that, for every $\varepsilon > 0$, the numbers r_k can be chosen so that

$$M(\Delta(E, \partial \mathbf{H})) < \varepsilon.$$

(c) Assume that the numbers r_k have been chosen as in (b) with $\varepsilon = 1$. Show that

$$M(\Delta(E_r, \partial \mathbf{H})) \rightarrow 0$$

when $r \rightarrow 0$ where $E_r = E \cap \overline{\mathbf{B}^n(r)}$.

5. Let $E, F \subset R(2, 1)$, $R(a, b) = \mathbf{B}^n(a) \setminus \overline{\mathbf{B}^n(b)}$, $a > b > 0$, and

$$\delta = M(\Delta(E, F, \mathbf{R}^n)) > 0.$$

Find a number $c > 1$, $c = c(n, \delta)$, such that $M(\Delta(E, F, R(2c, 1/c))) \geq \delta/2$.

6. Let $E \subset \mathbf{B}^n$ and $\delta = M(\Delta(S^{n-1}(2), E, \mathbf{R}^n)) > 0$. Find a number $c > 1$, $c = c(n, \delta)$, such that

$$M(\Delta(S^{n-1}(2), E, R(2c, 1/c))) \geq \delta/2.$$

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