Introduction to Conformal Geometry and Quasiconformal Maps
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Exercise 2, 2011-01-31 `File: icg1102.tex, 2011-1-23,20.53`
N.B. Numbered results/ formulas refer to CGQM.

1. Let f be an inversion in $S^{n-1}(a,r)$ as defined in 1.2(2)[CGQM]. Show that $f^{-1}=f$ and that $|x-a||f(x)-a|=r^2$ for all $x\in \mathbf{R}^n\setminus\{a\}$. By considering similar triangles show that the following identity holds for $x,y\in\mathbf{R}^n\setminus\{a\}$:

$$|f(x)-f(y)|=\frac{r^2|x-y|}{|x-a||y-a|}\ .$$

2. (a) For $0<t<1$ let $w(t)=t/\sqrt{1-t^2}$. Show that $q(0,w(t)e_1)=t$ and that

$$\frac{t}{s}<\frac{w(t)}{w(s)}<\frac{2t}{s}$$

for $0<s<t<\frac{1}{2}\sqrt{3}$.
(b) Let $q(A)=\sup\{q(x,y)\colon x,y\in A\}$ for $A\subset\overline{\mathbf{R}^n}$. Show that

$$q(Q(z,r))=q(\partial Q(z,r))=2r\sqrt{1-r^2}$$

for $0<r\le 1/\sqrt{2}$.

3.(a) Let $x,y\in\mathbf{B}^n$ with $s=q(0,x)$, $t=q(0,y)$. Show that

$$q(x,y)\ \le\ s\sqrt{1-t^2}+t\sqrt{1-s^2}\ \le\ t+s\ .$$

(b) Let $x,y\in\mathbf{R}^n\setminus\{0\}$ with $q(0,x)>q(0,y)$. Show that the strict inequality $q(x,y)>q(0,x)-q(0,y)$ holds.
4. For $x,y\in\mathbf{R}^n$ prove the following:

$$q(x,y)=\frac{|x-y|}{\sqrt{(1+|x||y|)^2+(|x|-|y|)^2}}.$$

$$\frac{|x-y|}{\sqrt{|x-y|^2+(1+|x||y|)^2}}\le q(x,y)\le\frac{|x-y|}{\sqrt{|x-y|^2+(1-|x||y|)^2}}.$$

$$q(x,y)\le\frac{|x-y|}{2}$$

for $|x||y|\ge 1$, with equality for $x,y\in S^{n-1}$ or $x=y$.

$$q(x,y)\le |x-y|/(|x|+|y|),$$

with equality iff $|x||y| = 1$ or $x = y$.

5. Let $h(x) = x/|x|^2$. Show that h maps the sphere $S^{n-1}(be_1, s)$ (assume that $b > 1 + s$) onto a sphere. Hint. Write $u = (b - s)e_1, v = (b + s)e_1$. If the image is a sphere $S^{n-1}(c, t)$, then clearly $c = (h(u) + h(v))/2$ and $t = |h(u) - h(v)|/2$. Hence it remains to show that $|z - be_1| = s$ implies $|h(z) - c| = t$.

6. The lines $[-e_1, 0]$ and $[ae_1, \infty]$, $a > 0$, can be mapped onto $[-e_1, e_1]$ and $[be_1, \infty] \cup [-be_1, \infty]$ by a Möbius transformation. Give a definition for b in terms of a. Notice that $[x, \infty] = \{xt : t \geq 1\}$, if $x \in \mathbf{R}^n \setminus \{0\}$.