### The model of Gause

\[
\begin{align*}
\frac{dR}{dt} &= f(R) - g(R)X \quad \text{(resource)} \\
\frac{dX}{dt} &= Yg(R)X - \delta X \quad \text{(consumer)}
\end{align*}
\]

**Assumptions**

1. \(f, g\) continuously differentiable.
2. \(f(0) = 0\), and \(\exists K > 0 : f(R) > 0\) for \(0 < R < K\) and \(f(R) < 0\) for \(R > K\).
3. \(g(0) = 0\), and \(g(R), g'(R) > 0\) for \(R > 0\).
4. \(\exists R_0 > 0 : Yg(R_0) = \delta\). (unique solution)

**Remarks**

- Resource dynamics without consumer then a globally attracting equillibrium \(R = K\).
- By monotony of \(g\), there exist only one \(R_0 > 0\) such that \(Yg(R_0) = \delta\).
- If there were no such \(R_0 > 0\), then \(Yg(R) < \delta \forall R > 0\) (since \(g(0) = 0\)), so that \(X \rightarrow 0\) as \(t \rightarrow \infty\), and hence also \(R \rightarrow K\) as \(t \rightarrow \infty\).
- \(g(R)\) is called the functional response of the consumer.
Phase plane analysis.

\[ \frac{dR}{dt} = 0 \Leftrightarrow R = 0 \text{ or } X = \frac{f(R)}{q(R)} \]
\[ \frac{dX}{dt} = 0 \Leftrightarrow X = 0 \text{ or } R = R_0 \]

**Case \( R_0 \geq k \)**

\[ \]

- Since \( \frac{f(R)}{q(R)} < 0 \) for \( R = R_0 \), there exists no positive equilibrium.
- The point \((k,0)\) is globally stable.

To see this, first notice that \( R(t) \) is monotonically decreasing on the right side of the line \( R = R_1 \), while on the left side \( X(t) \) is monotonically decreasing. Thus, every orbit with a positive starting point eventually enters the set \( D_\delta = (0,R_1) \times (0,\delta) \), where \( \delta > 0 \) can be taken arbitrarily small.

Inside \( D_\delta \), orbits eventually enter \((k-\delta, k+\delta) \times (0,\delta)\). We can choose \( \delta > 0 \) arbitrarily small provided we adjust \( \delta > 0 \) accordingly.
Case \( 0 < R_0 < K \)

- There exists a unique positive equilibrium \((\hat{R}, \hat{x})\) with \(\hat{R} = R_0\) and \(\hat{x} = \frac{g(R_0)}{g'(R_0)}\).

- The boundary equilibria \((0,0)\) and \((K,0)\) are obviously unstable.

Local stability of \((\hat{R}, \hat{x})\):

Jacobi-matrix of \(f\) (p.71) evaluated at \((\hat{R}, \hat{x})\):

\[
J = \begin{pmatrix}
g(\hat{R}) - \hat{x}g'(\hat{R}) & -g'(\hat{R}) \\
\hat{x}g'(\hat{R}) & \hat{x}g''(\hat{R}) - \delta
\end{pmatrix} = \begin{pmatrix}
g(\hat{R}) & \frac{g(\hat{R})}{g'(\hat{R})}' - g'(\hat{R}) \\
\hat{x}g'(\hat{R}) & 0
\end{pmatrix}
\]

\[
det J = \hat{x}g'(\hat{R})g(\hat{R}) > 0
\]

\[
\text{trace } J = g(\hat{R}) \left[ \frac{g(\hat{R})}{g'(\hat{R})}' \right] > 0
\]
Conclusion (see Appendix A, p. A10)

The equilibrium \((k, 0)\) is a saddle.

Let \(p\) be a point in \(R^2\) on the unstable manifold of the saddle.

Obviously, the forward orbit through \(p\) is bounded.

By the Poincaré–Bendixson theorem (see Appendix B), there are two alternatives.

1. The \(\omega\)-limit contains the point \((\hat{k}, \hat{x})\),
   (which then must be globally stable.)
6. The $\omega$-limit is a closed orbit (limit-cycle) which must circle the equil.

Obviously, in case of we must have that $(\tilde{R}, \tilde{x})$ is stable.

From Poincaré-Bendixson it also follows that if $(\tilde{R}, \tilde{x})$ is unstable, then there must exist a limit cycle.

If $(\tilde{R}, \tilde{x})$ is unstable, it is either a node or a focus, but not a saddle (det $J > 0$, p. 73).

Then we can construct an annular forward invariant region $D$ that contains no equilibria.

It follows that $D$ contains a limit cycle (→ appendix B, p. B2).
\[
\begin{align*}
\frac{dx}{dt} &= \nu R \left(1 - \frac{R}{K}\right) - \frac{BR}{1 + \rho TR} \cdot X \\
\frac{dR}{dt} &= \gamma \cdot \frac{BR}{1 + \rho TR} \cdot X - SX
\end{align*}
\]

Suppose \( \frac{S}{\gamma} < \frac{1}{\nu} \). Then it is a special case of the Gauss model with

\[
g(R) = \nu R \left(1 - \frac{R}{K}\right)
\]

and

\[
g(R) = \frac{BR}{1 + \rho TR}
\]

0 < \( R_0 < \frac{1}{2} \left( K - \frac{1}{\nu \rho} \right) \)

\( \frac{1}{2} \left( K - \frac{1}{\nu \rho} \right) < R_0 < K \)

[\( \frac{\delta}{\gamma} \) > 0 at \((\hat{R}, \hat{K})\) \( \Rightarrow \)

\((\hat{R}, \hat{K})\) is unstable and there exist a limit cycle

[\( \frac{\delta}{\gamma} \) < 0 at \((\hat{R}, \hat{K})\) \( \Rightarrow \)

\((\hat{R}, \hat{K})\) is locally stable]
Proposition.

Let \( \frac{\delta}{\theta} < \frac{1}{\alpha} \) and \( \alpha^{-\frac{1}{\alpha}} (K - \frac{1}{\alpha}) \leq K_0 < K \). Then \((R, \tilde{x})\) is globally stable.

Proof.

By Poincaré–Bendixson’s theorem, it is sufficient to show that there cannot exist limit cycles. For this, we use the criterion of Dulac (see Appendix B, p. 85).

Consider the Dulac function

\[
U(R, x) = \frac{x^{\alpha-1}}{q(R)}
\]

where \( \alpha > 0 \) will be chosen later.

\[
\begin{align*}
\text{div} \left( \begin{array}{c}
U \frac{dR}{dt} \\
U \frac{dx}{dt}
\end{array} \right) &= \\
&= x^{\alpha-1} \left( \left[ \frac{q(R)}{q(R)} \right]' + \alpha \left( \beta - \frac{1}{\alpha} \right) \right) \\
&= R^\gamma x^{\alpha-1} \left( \beta R (\alpha T - \frac{1}{\alpha} - \frac{2bT}{K} R) - \alpha \left( \frac{\delta}{\theta} - \frac{1}{\gamma} \right) R \right)
\end{align*}
\]

The graph of \( q(R) \) is a parabola.
Obviously, $\alpha > 0$ can be chosen such that the straight line always lies above the parabola.

(For example, we can choose $\alpha$ such that the slope of the line is equal to the slope of the parabola at $R = \frac{1}{2}(K - \frac{1}{\alpha})$)

Then also $U_{axx} \leq 0$ for all $(R, x)$, with equality to zero only if $R = R_0$ (and $R_0 = \frac{1}{2}(K - \frac{1}{\alpha})$; see figure below.)

Hence, there exist no limit cycles.